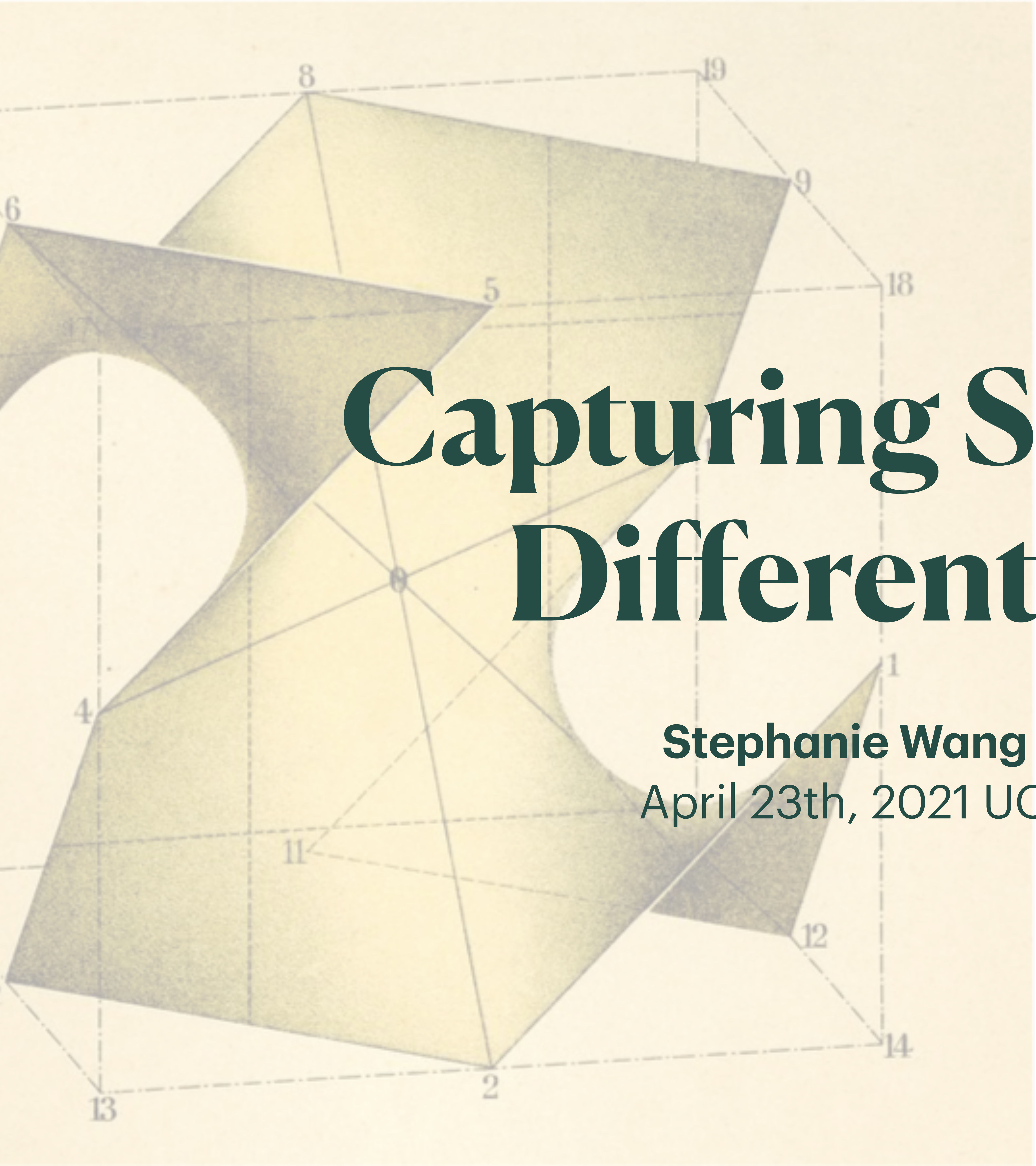
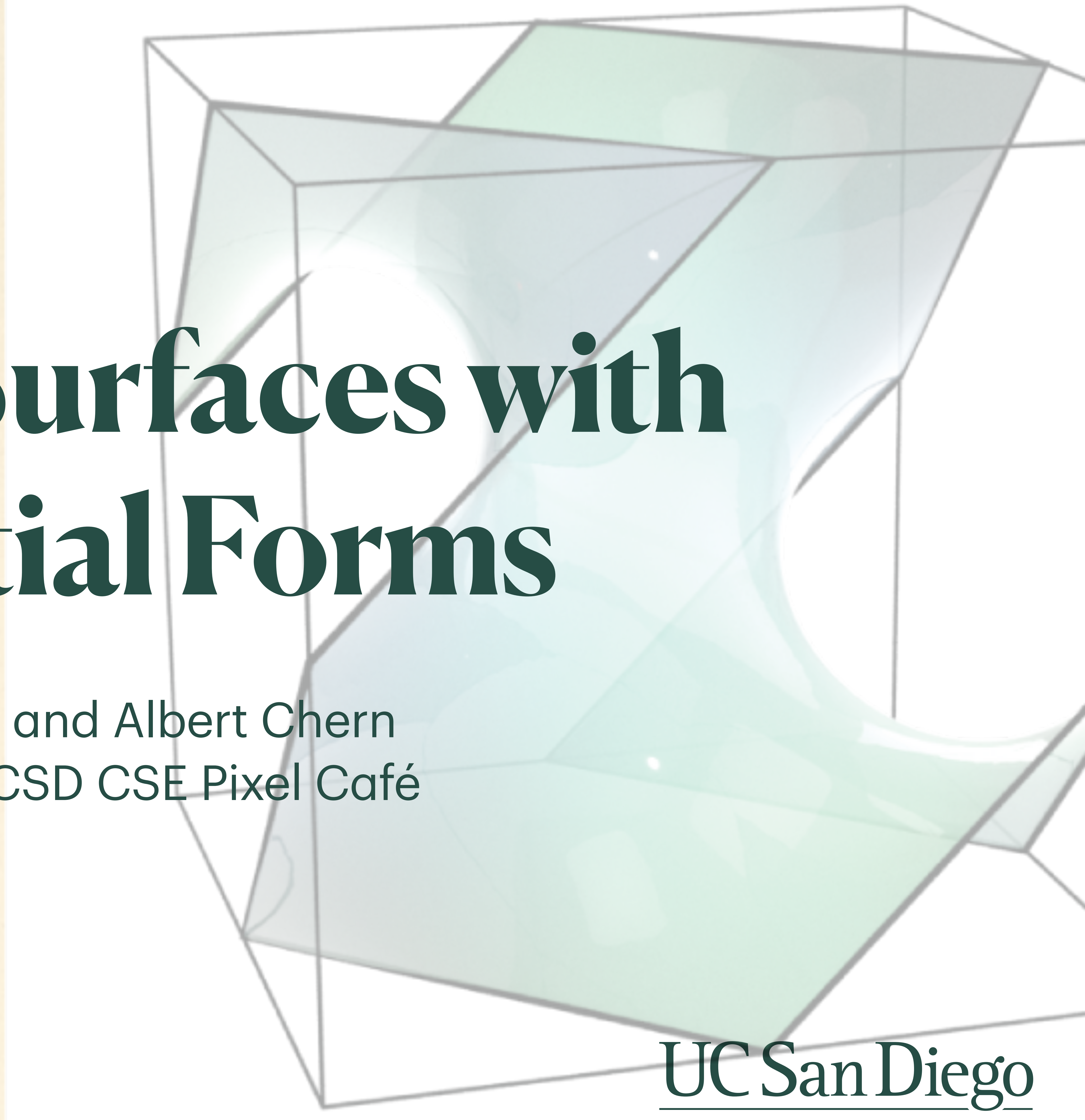




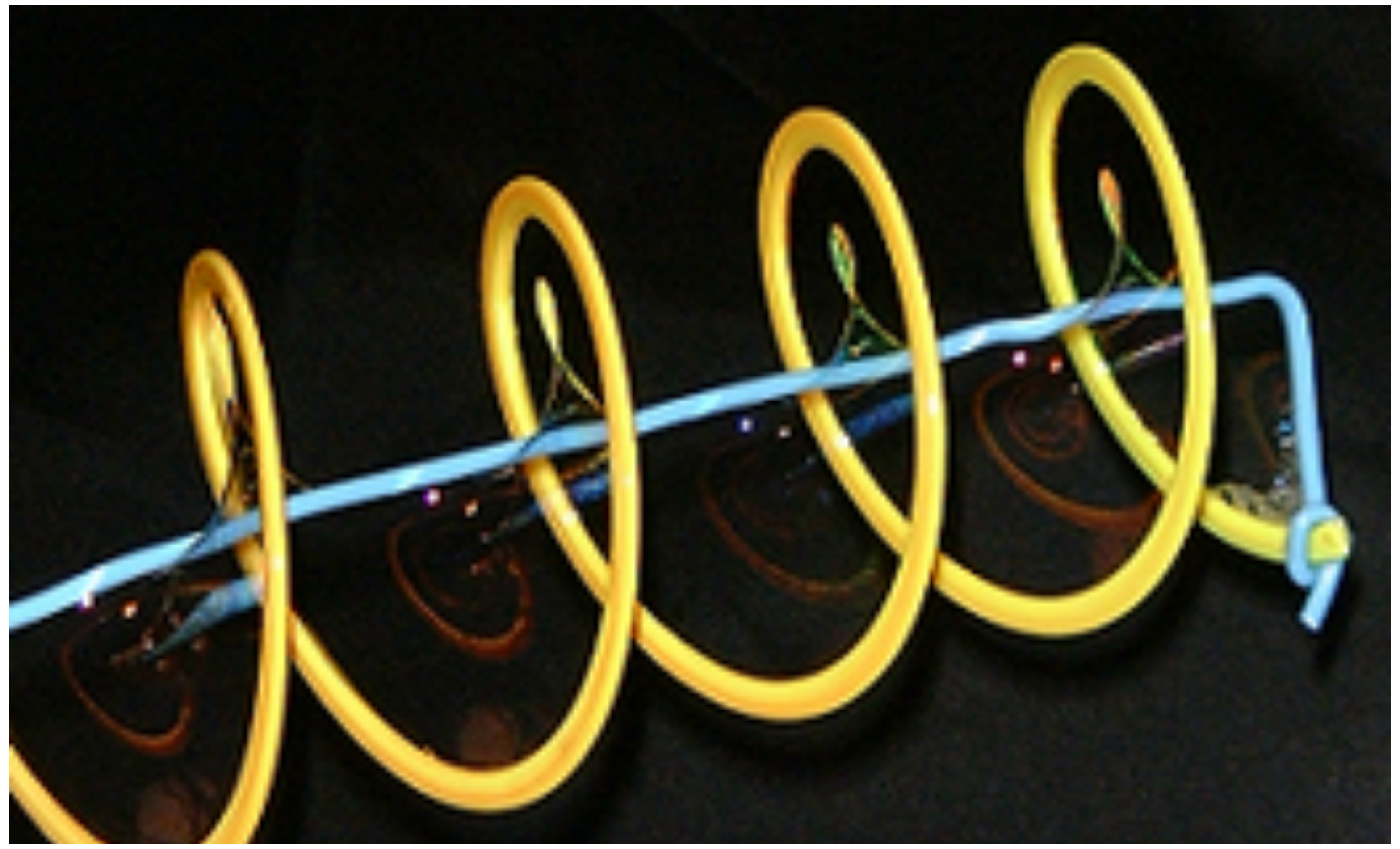
Capturing Surfaces with Differential Forms

Stephanie Wang and Albert Chern

April 23th, 2021 UCSD CSE Pixel Café

UC San Diego

Helicoid formed by soap film on a helical frame PC: Blinking Spirit



Münich Olympiapark PC: Tiia Monto

RUBATO by Eva Hild PC: David Eppstein

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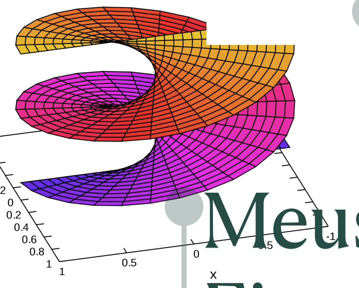
A brief history of minimal surfaces

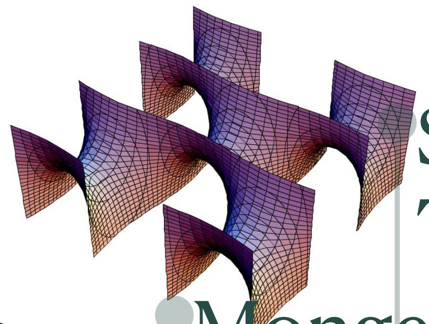
The chronicle

Lagrange 1760 $k + \left(\frac{d\mathrm{P}}{dx}\right) + \left(\frac{d\mathrm{Q}}{dy}\right) = 0,$
Euler-Lagrange equation of surface area functional for height field

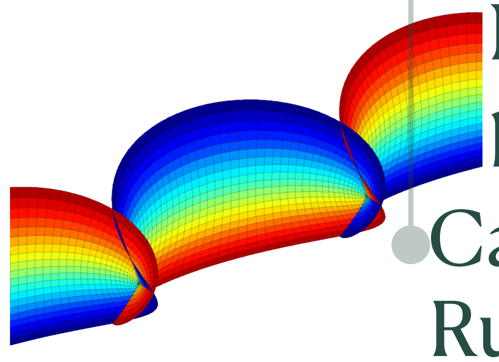


Meusnier 1776
First non-trivial example: helicoid and catenoid

Monge and Legendre 1795
Representation formulas for minimal surfaces

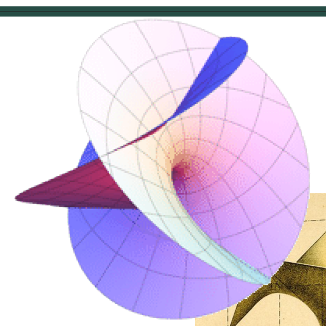


Scherk 1834
Third example of minimal surfaces (periodic)

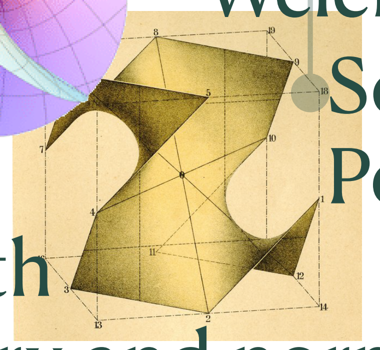


Catalan 1842-43
Ruled minimal surface (helicoid)

Björling 1844
Minimal surface with
prescribed boundary and normal



Enneper and Weierstrass 1863?
Weierstrass representation and Enneper surface

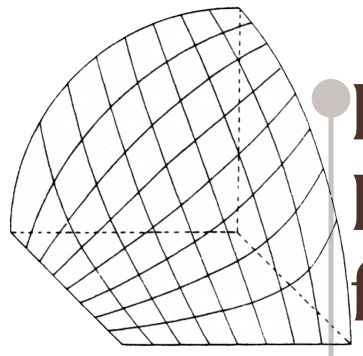


Schwarz 1865-1867
Periodic Schwarz surface families

Plateau 1873
Plateau's laws of soap films
(including nonmanifold soap films)

Kelvin 1887
Bubble foam packing problem

Bernstein 1914
Boundaryless function graphs in R^3 are planes



Douglas 1927
Finite difference with multigrid acceleration
for height field with arbitrary boundary value

Douglas 1931, Rado 1931 (independently)
Existence for disks based on conformal mapping

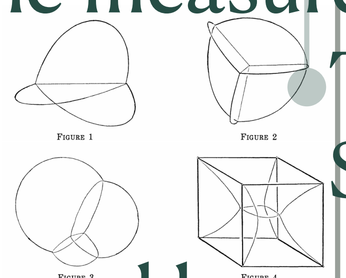
Courant 1936-40
Existence for more general boundary curves with con

Federer and Fleming 1960
Geometric measure theory

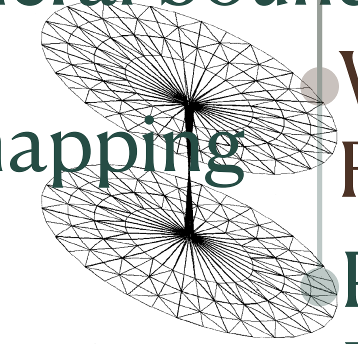
Wilson 1961
Finite difference with conformal map

Concus 1967
Finite difference for height f

Hinata, Shimasaki, and Ki
Finite element for height f

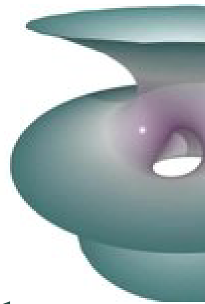


Taylor 1976
Soap film singularities



Wagner 1977
Finite element for ge

Parks 1977 (1986 boo
Levelset of functions
with boundary on bo



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A brief history of minimal surfaces

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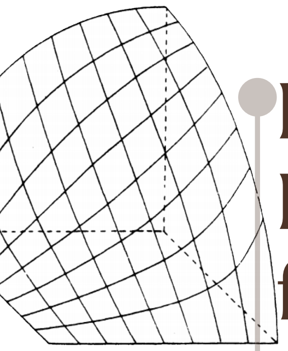

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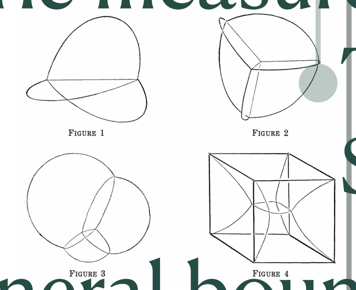

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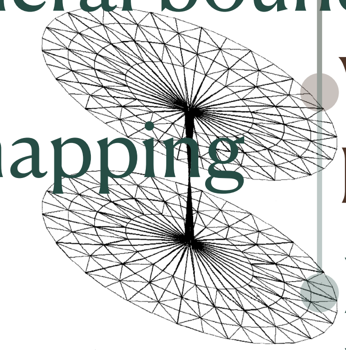
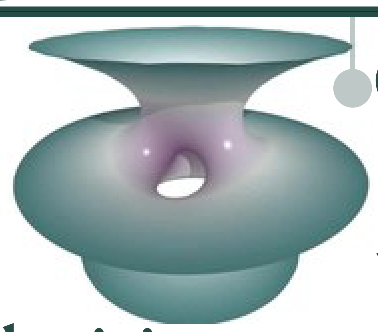
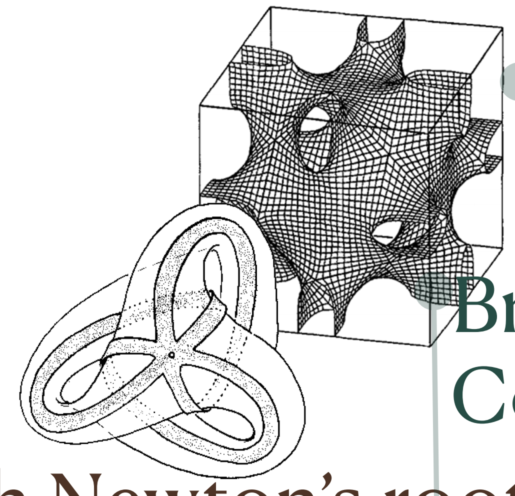


Wagner 1977
Finite element for general trimesh

Parks 1977 (1986 book)
Levelset of functions with minimal total variations
with boundary on boundary of convex open sets

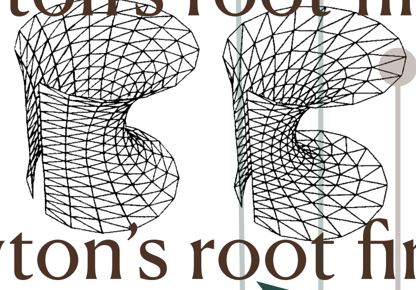


Costa 1982
Example of minimal surface other than
plane, catenoid, and helicoid

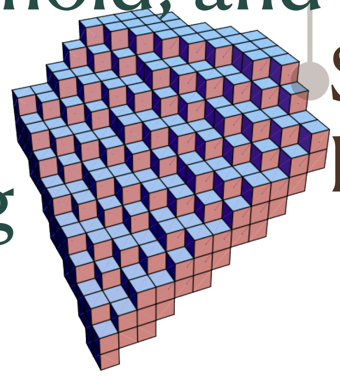


Schoen 1970, Karcher 1989
Triply periodic minimal surfaces (lattices)

Bryant 1987
Constant mean curvature surface in 3D hyperb



Dziuk 1990
Discrete mean curvature L2 flow



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Discrete minimal current with linear progr

Brakke 1992
Surface Evolver (software)

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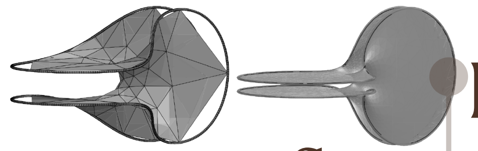
Implementation of his 1977 work



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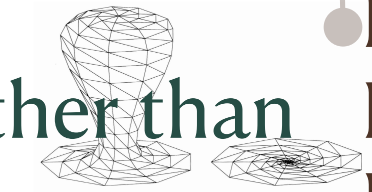
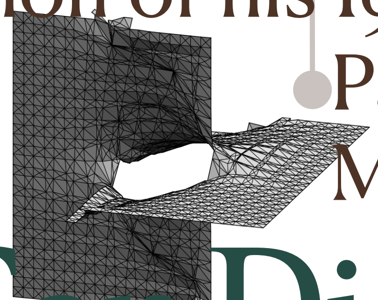


Pinkall and Polthier 1
Discrete mean curvat
H1 gradient flow

Popov 1996
Stretched grid met



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Minimal to

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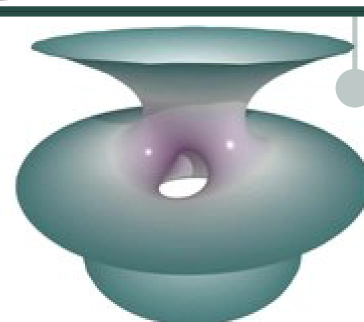

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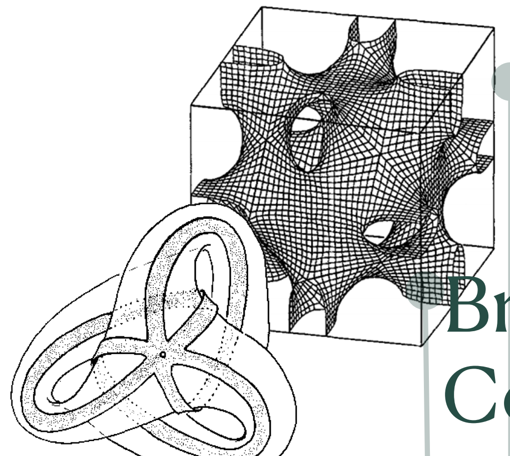

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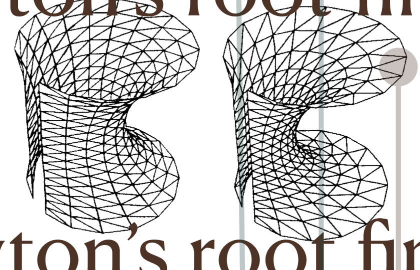

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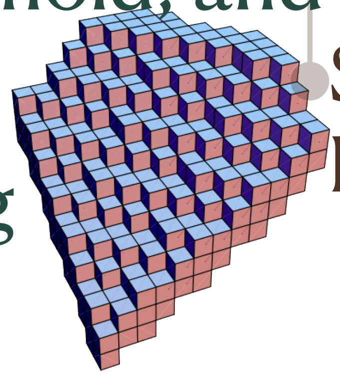

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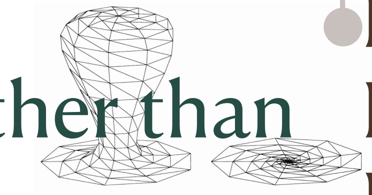

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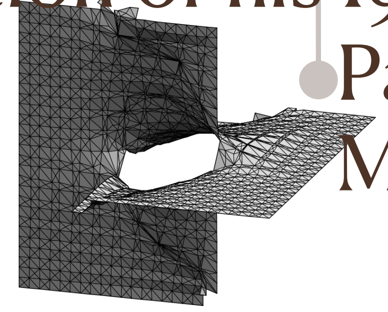
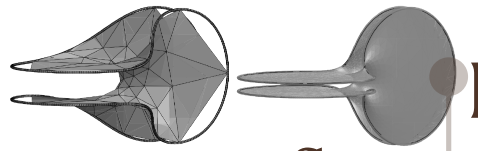
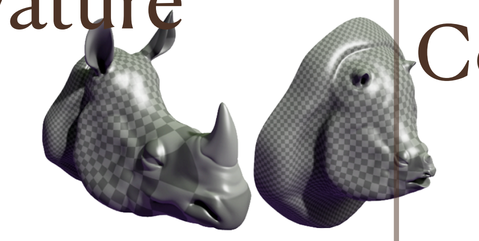


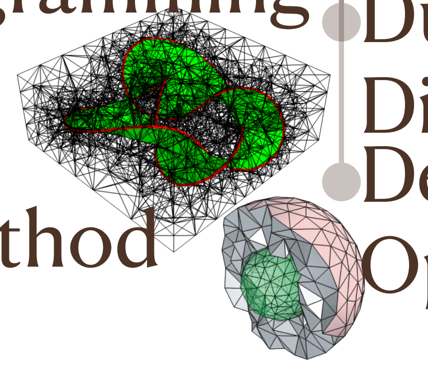
Parks and Pitts 1997
Minimal total variation levelset with branch cut



Dziuk and Hutchinson 1999
Convergence analysis on Plateau problem algorithms

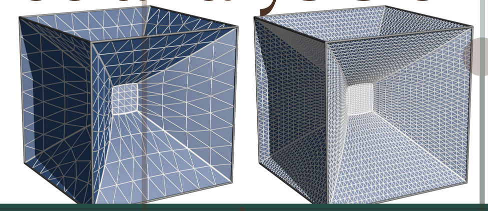


Crane, Pinkall, and Schröder 2011
Conformal Willmore flow minimal surface



Dunfield and Hirani 2011
Discrete minimal current on tetmesh

Dey, Hirani, and Krishnamoorthy 2011
Optimal homologous cycle



Kazhdan, Solomon, and Ben-Chen 2012
Conformal minimal surface by finite element heat flow

Brezis and Mironescu 2019
S1-valued minimizer of total variation

Schumacher and Wardetzky 2019
Convergenece under refinement

Parks and Pitts 2020
Discrete minimal current with Surface Evolver

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Plateau problem

finding the minimal surface with a given boundary curve

- M: a (3-dimensional) manifold.
- Given a closed boundary curve $\Gamma \hookrightarrow M$, that is, $\partial\Gamma = \varnothing$.
- Find a surface $\Sigma \hookrightarrow M$ that

$$\begin{aligned} \text{minimize} \quad & \mathrm{Area}(\Sigma) \\ \text{s.t.} \quad & \partial\Sigma = \Gamma \end{aligned}$$

“Area functional is not convex.”

—well-known geometric processing fact.

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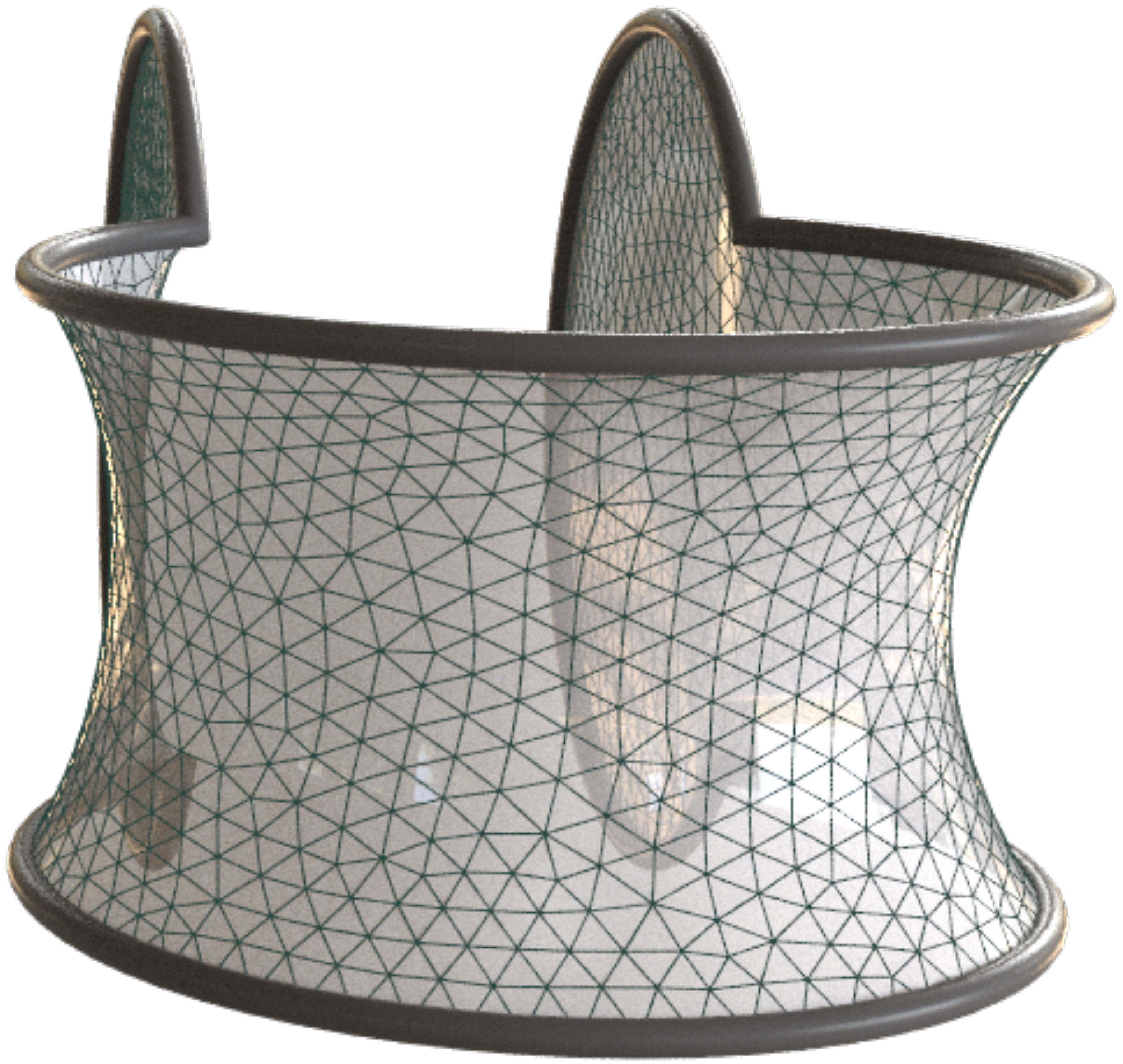

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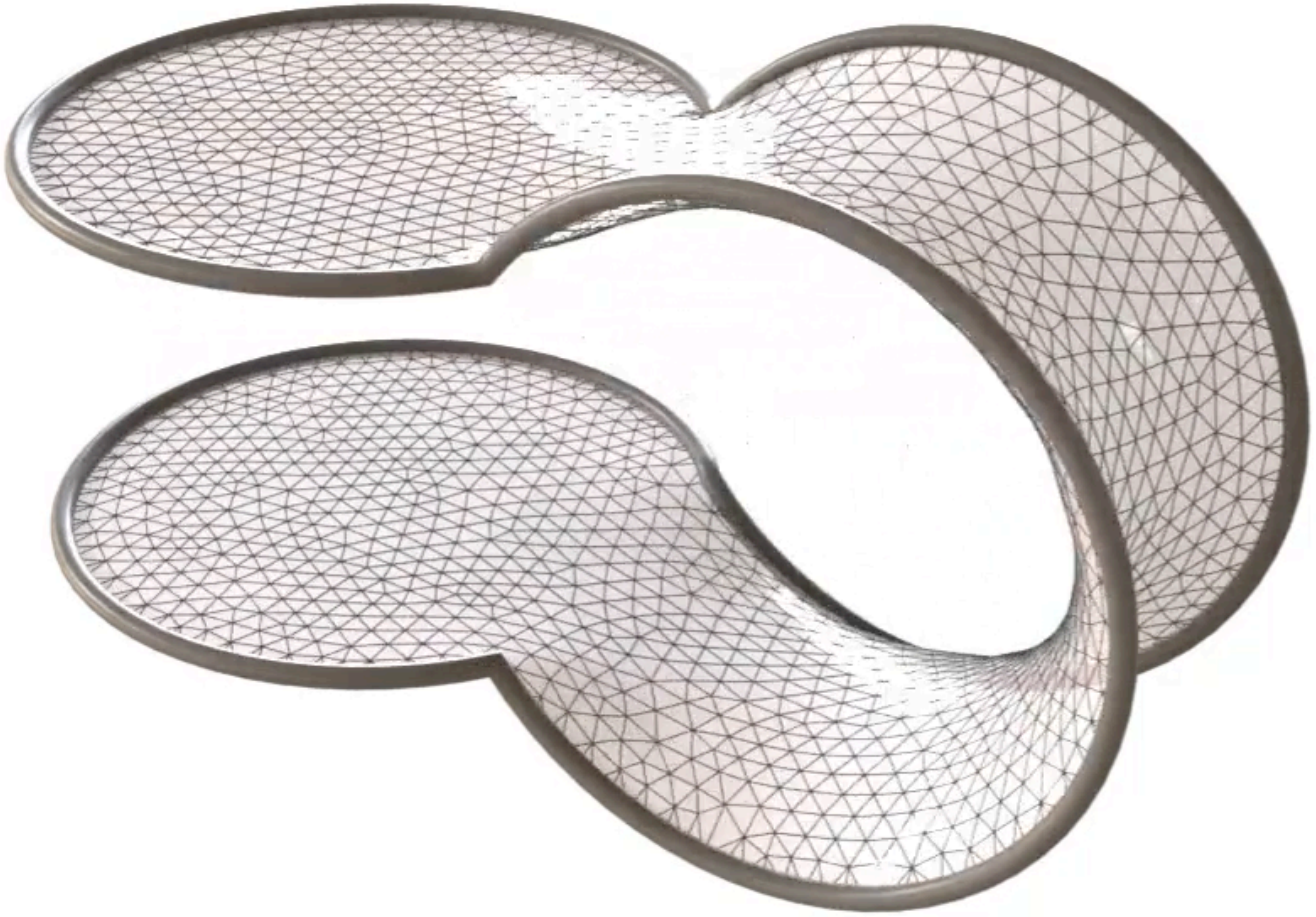
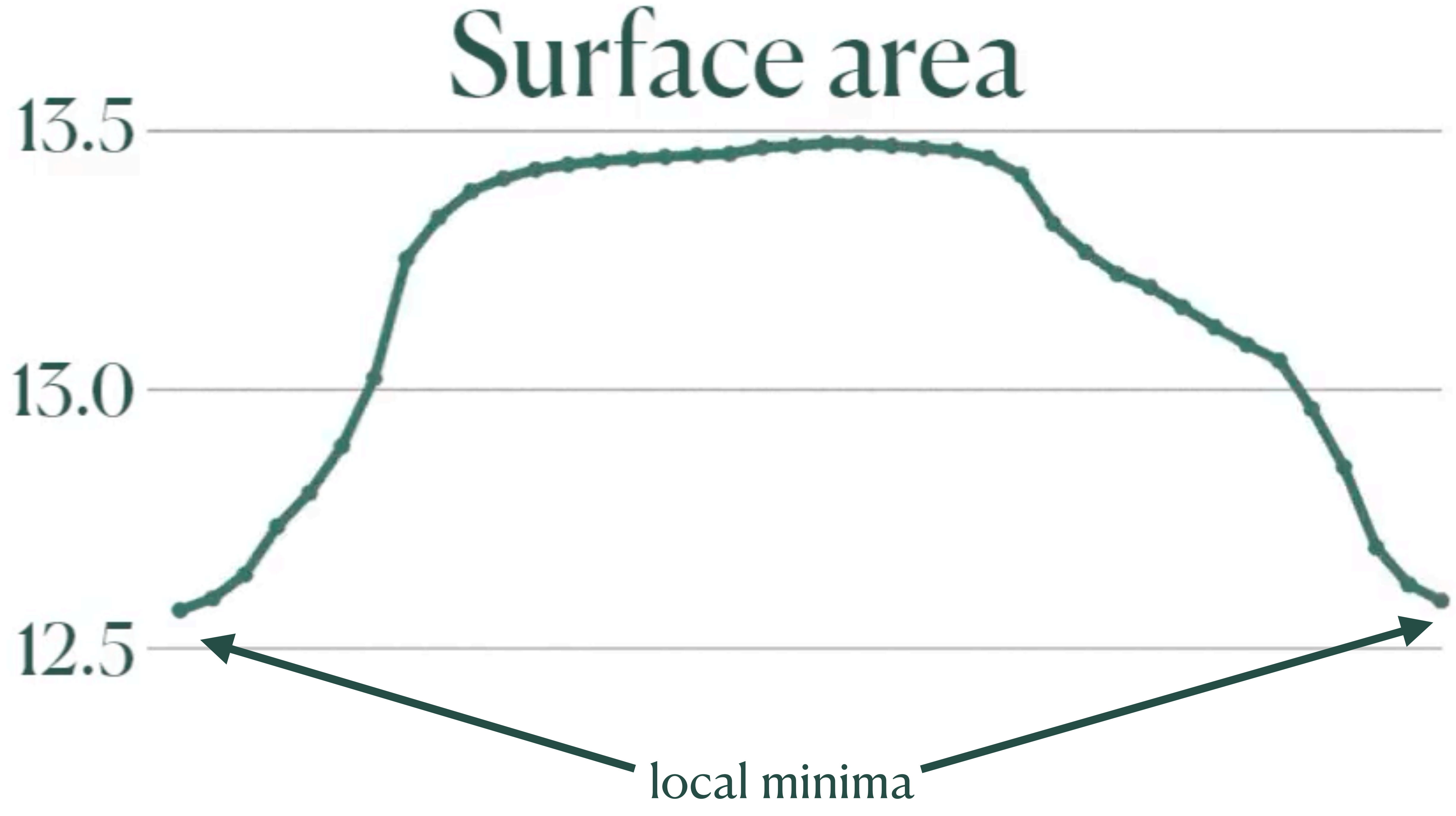
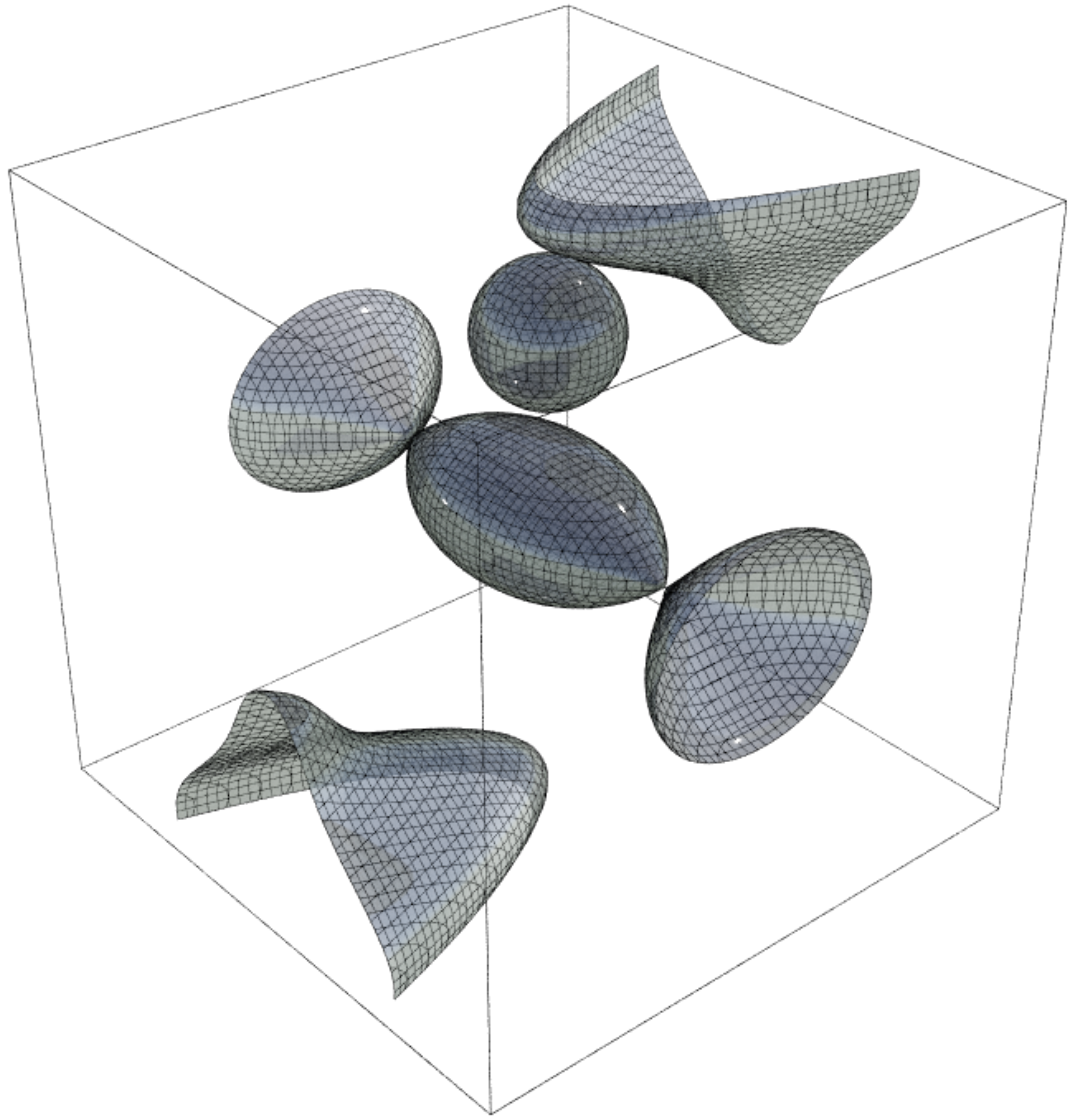


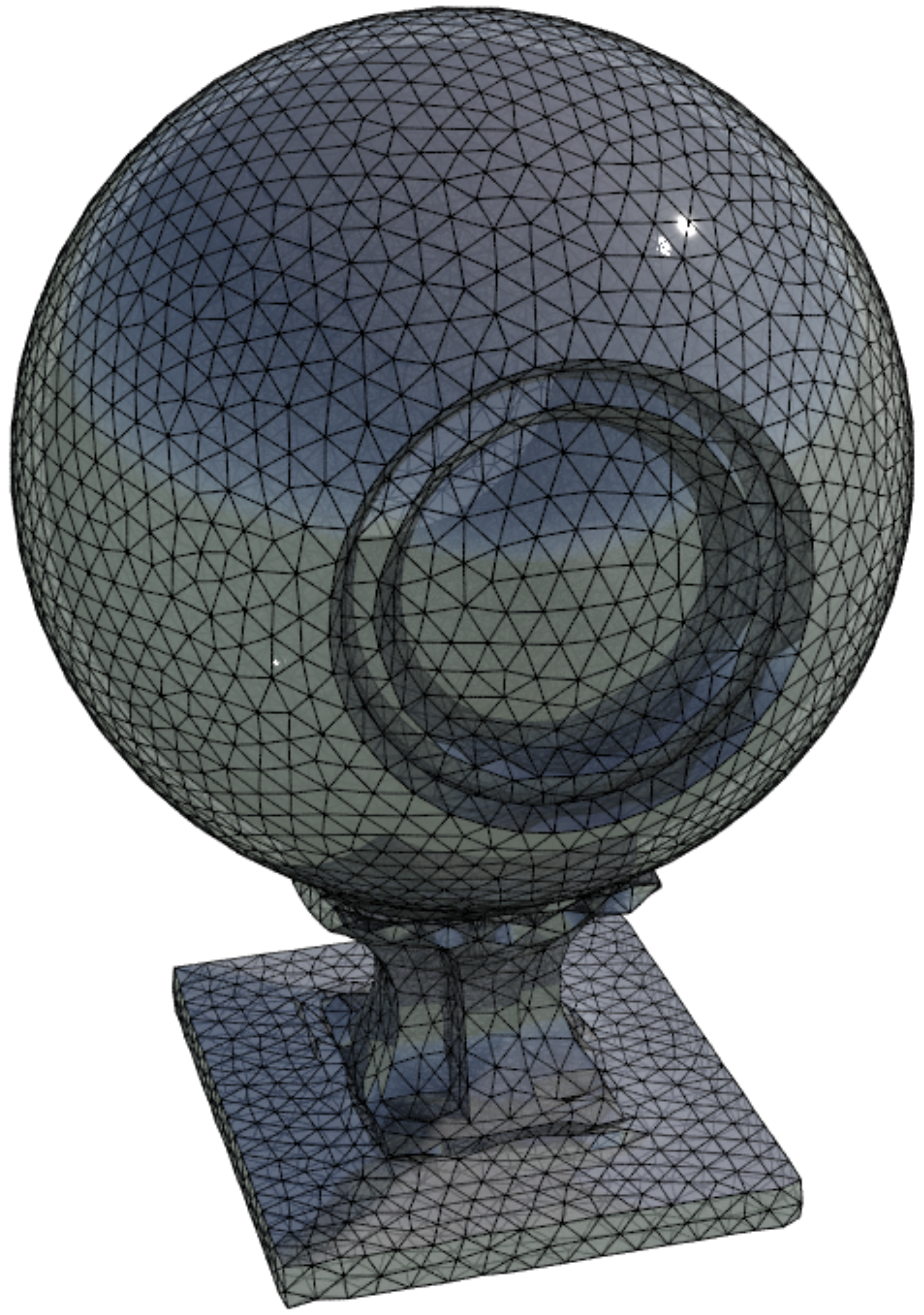


Surface representations

Implicit



Explicit



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Level-set-based methods

level sets are always implicitly closed

- Surface is implicitly stored with a spatial function $f : M \to \mathbb{R}$
  - Level set $\Sigma = \{\mathbf{x} \in M | f(\mathbf{x}) = 0\}$

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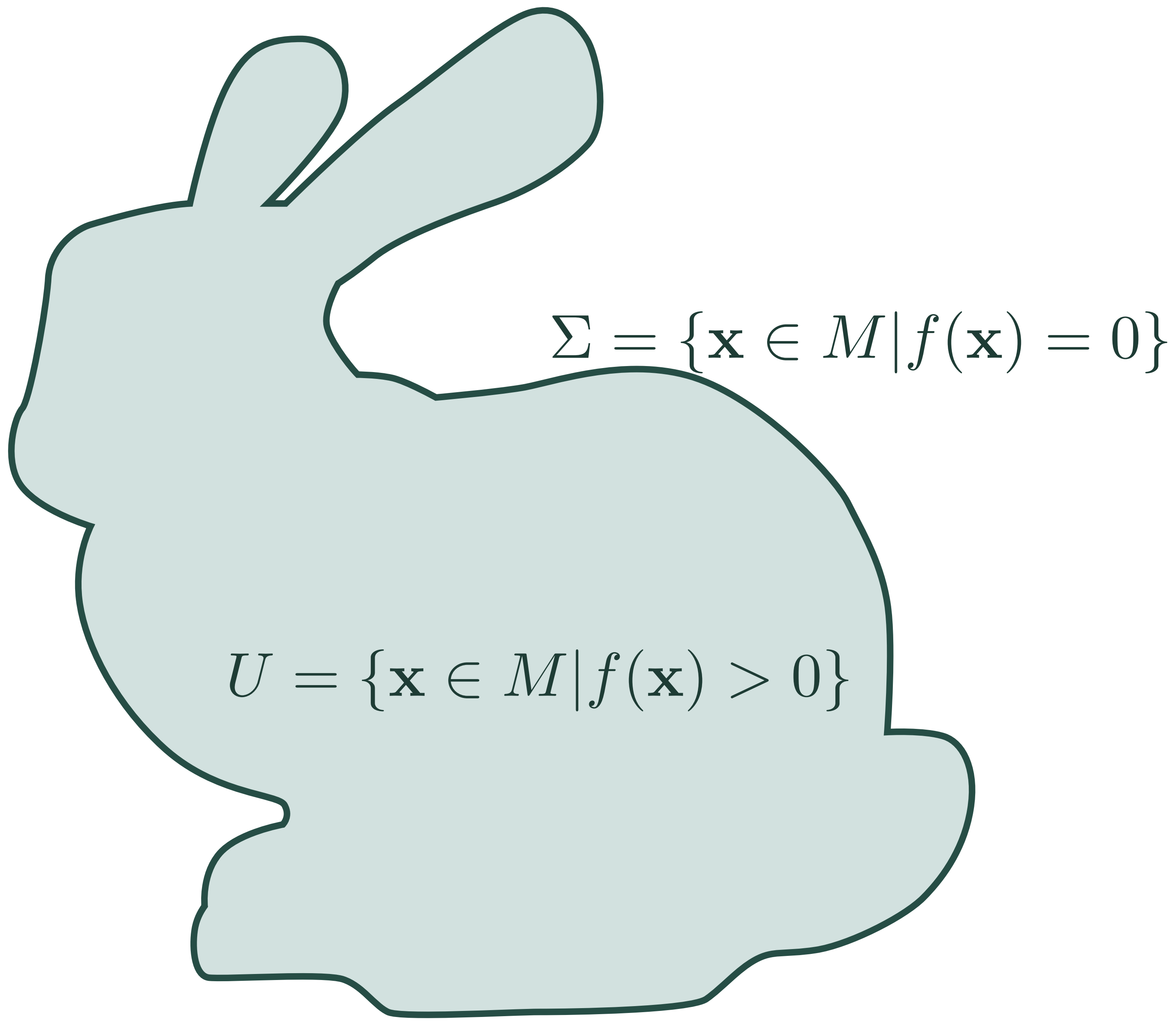
- Inherently, $\Sigma = \partial\{\mathbf{x} \in M | f(\mathbf{x}) > 0\} = \partial U$ boundary of an open region.
  - $\partial\Sigma = \partial\partial U = \varnothing$.



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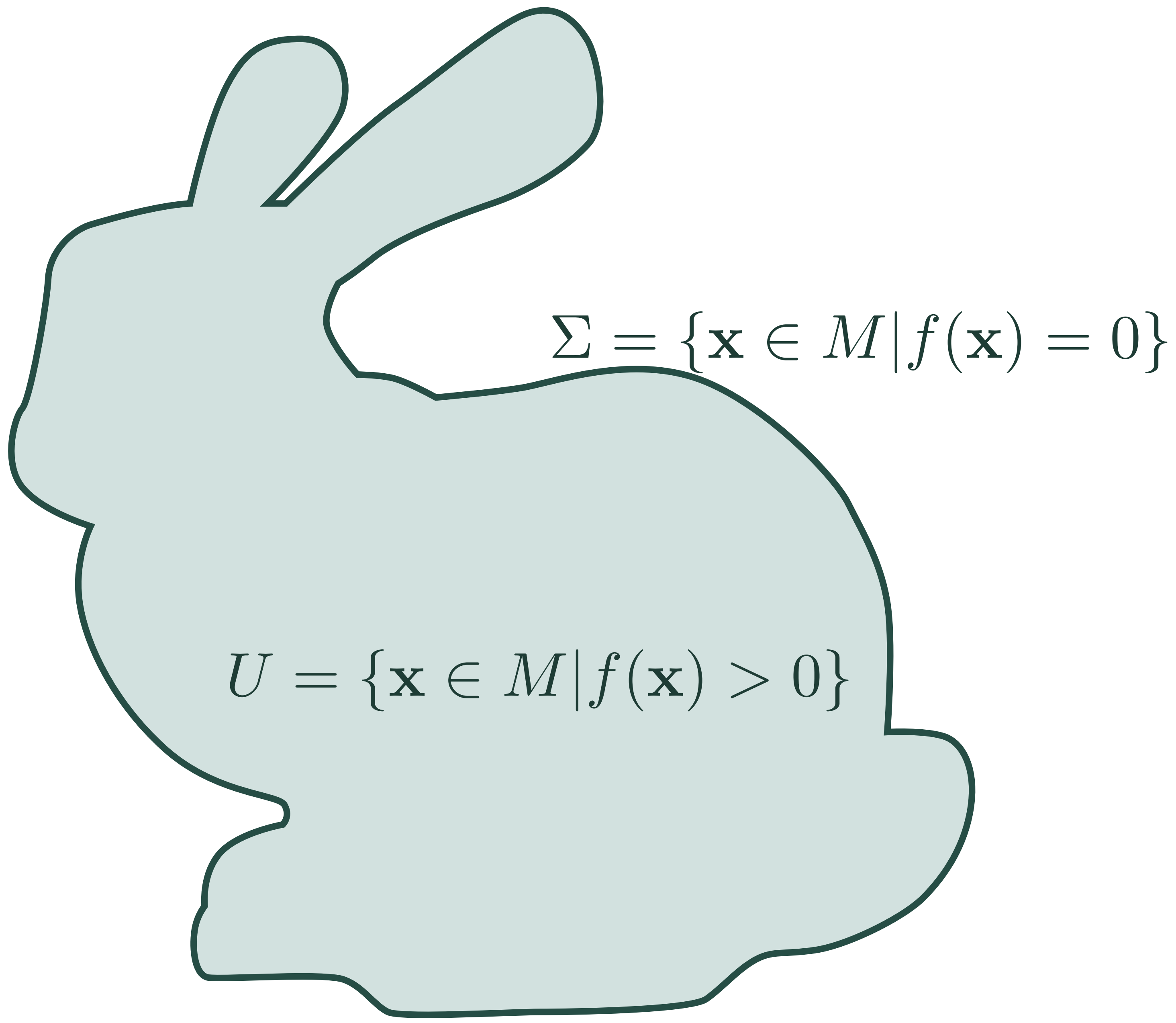
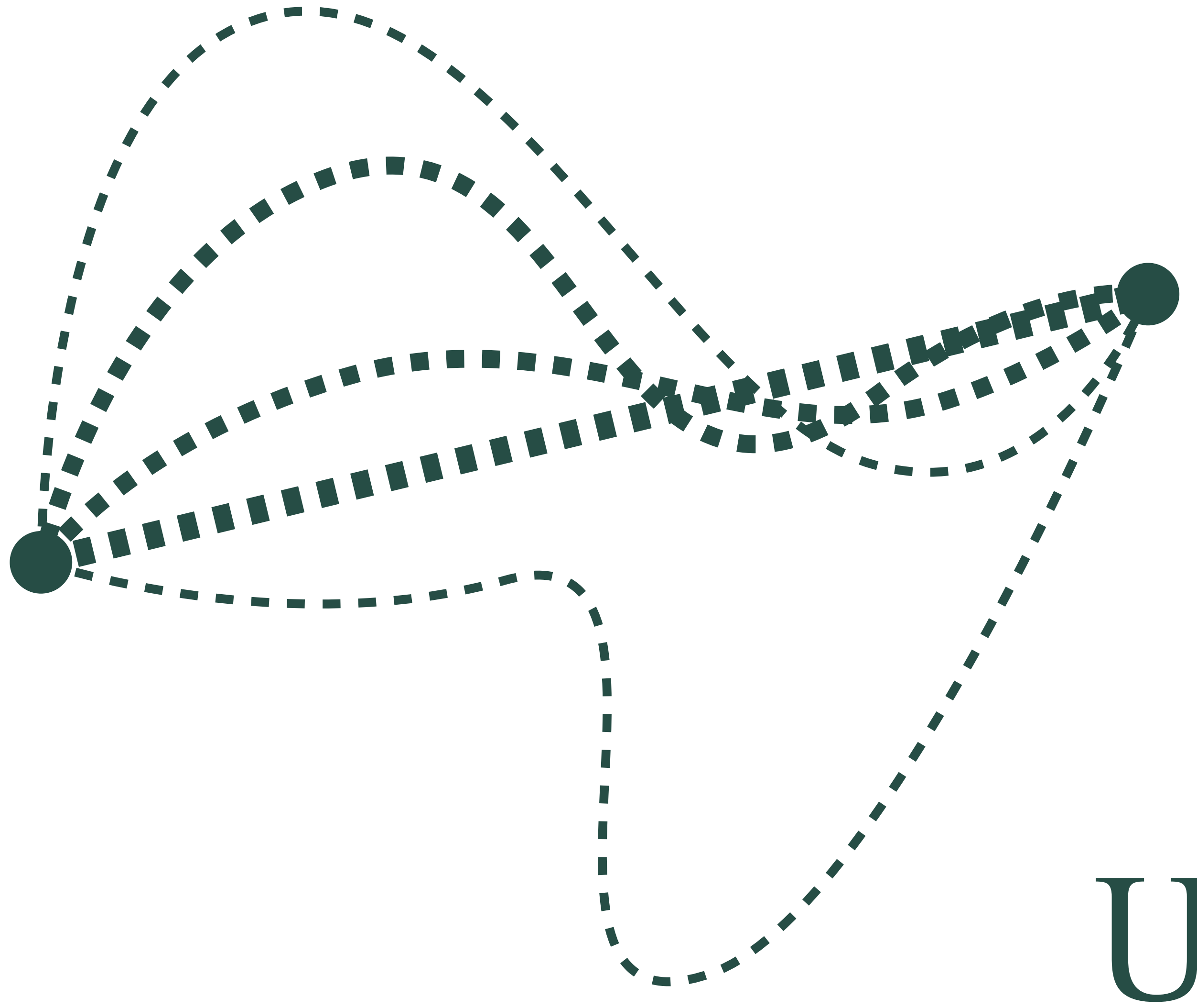




Mesh-based methods

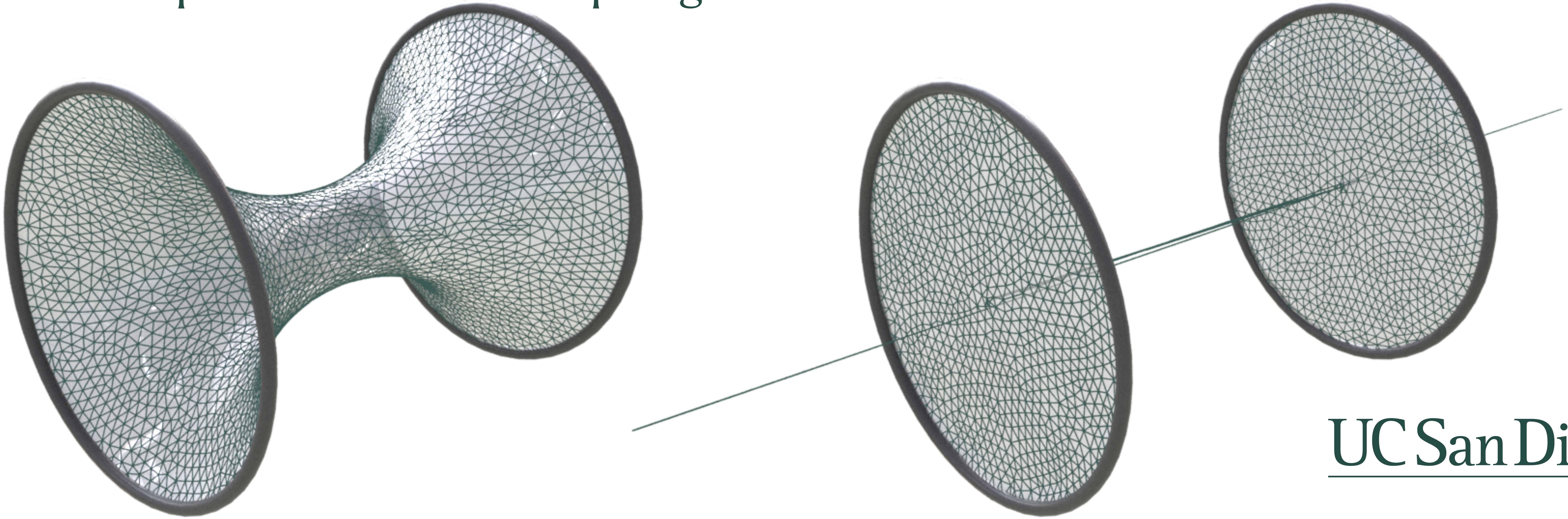
mesh-quality-dependent Laplacian

- Discrete curvature flow
- Ill-conditioned Laplacian for low quality mesh

Mesh-based methods

explicit representation means discrete connectivity

- Discrete curvature flow
- Ill-conditioned Laplacian for low quality mesh
- Discrete operation to resolve topological difference



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Our method

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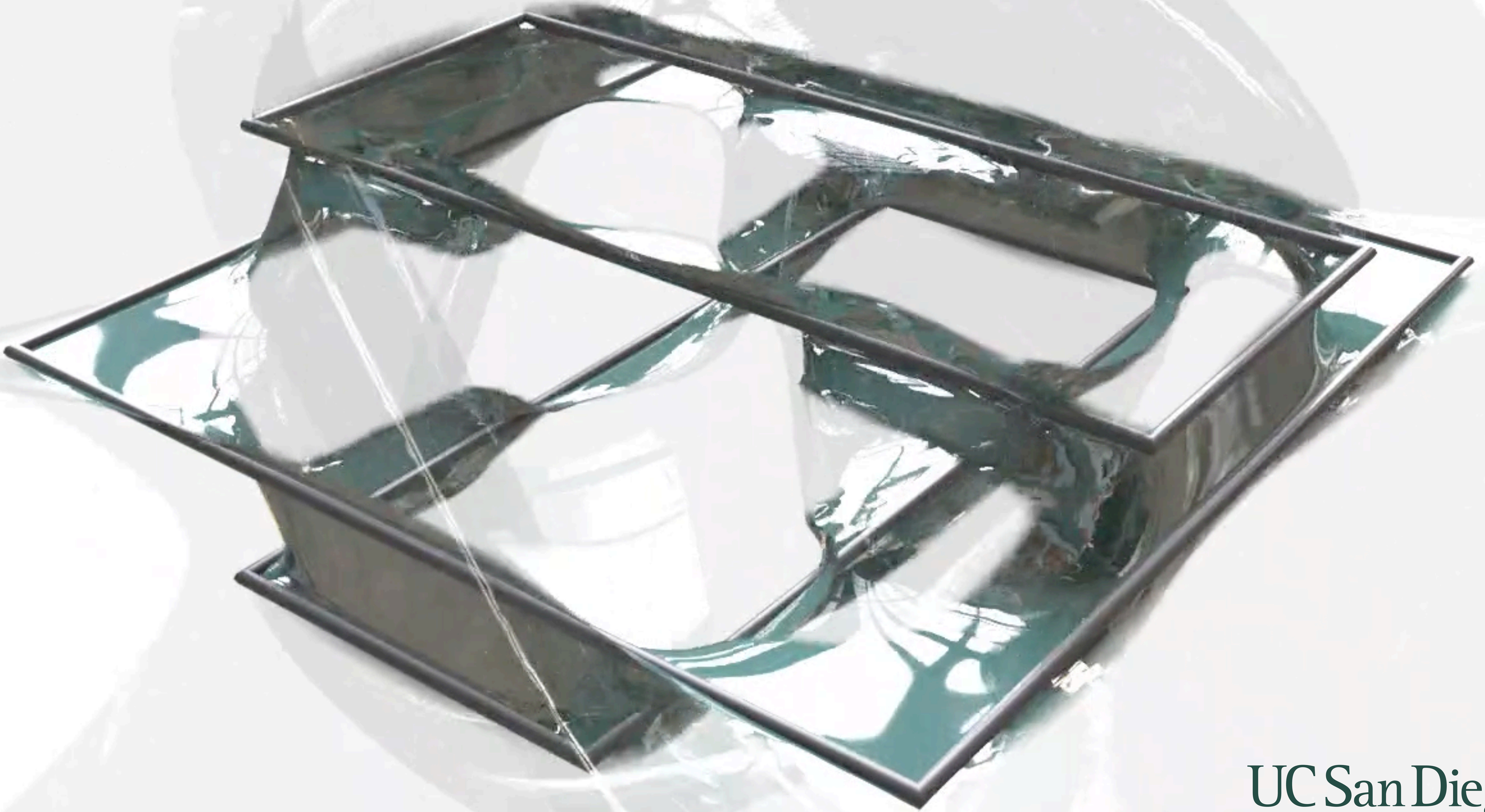

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Outline

- Current = Differential form & distribution
- Boundary constraint = weak derivative constraint of current
- Area functional = mass norm of current
- Cohomology condition
- ADMM and results

Problem setup

(the assumptions)

- The "manifold" $M = \mathbb{T}^3$
  - (we chose compact M intentionally)
- The "boundary curve" $\Gamma \hookrightarrow M$
  - A closed curve, i.e. $\partial\Gamma = \varnothing$

Differential forms

c.f. Bott & Tu 1991

- $\Omega^* = \{1, dx_1, dx_2, dx_3, dx_1 \wedge dx_2, \cdots, dx_1 \wedge dx_2 \wedge dx_3\}$ with relation

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

- Differential forms on M is

$$\Omega^*(M) = C^\infty(M) \underset{\mathbb{R}}{\otimes} \Omega^* = \bigoplus_{k=0}^{3} \Omega^k(M)$$

- Exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \qquad d(f_I dx_I) = df_I \wedge dx_I$$

Integrals

natural linear functional on differential forms

- Natural pairing between k forms and k-submanifolds

$$\langle \Sigma | \omega \rangle = \int_\Sigma \omega \qquad \int_S f(x,y)\,dx\,dy = \int_S f\,dA$$

- Integral is linear in ω,

$$\left(\Omega^k(M)\right)^* \supset \left\{ \omega \mapsto \int_\Sigma \omega \;\middle|\; \Sigma \hookrightarrow M\colon\ k\text{-submanifold} \right\}$$

$$\int_{\alpha_1 S_1 + \alpha_2 S_2} f(x,y)\,dx\,dy = \alpha_1 \int_{S_1} f\,dA + \alpha_2 \int_{S_2} f\,dA$$

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Integrals

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$$\boxed{\{\Sigma : k\text{-dim geometry}\} \longleftrightarrow \left\{ \omega \mapsto \int_\Sigma \omega : \text{integrals of } k\text{-forms} \right\}}$$

Dirac-delta ~~function~~ measure

0-dim geometry (points) $\longleftrightarrow$ dual of 0-forms (functions)

- A point $p \in M$ can be represented by a Dirac-delta measure

$$\delta_p = \begin{cases} \infty \text{ at } p \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} \delta_p : C^\infty(M) &\to \mathbb{R} \\ f &\mapsto f(p) \end{aligned}$$

$$\int_M f\delta_p = f(p)$$

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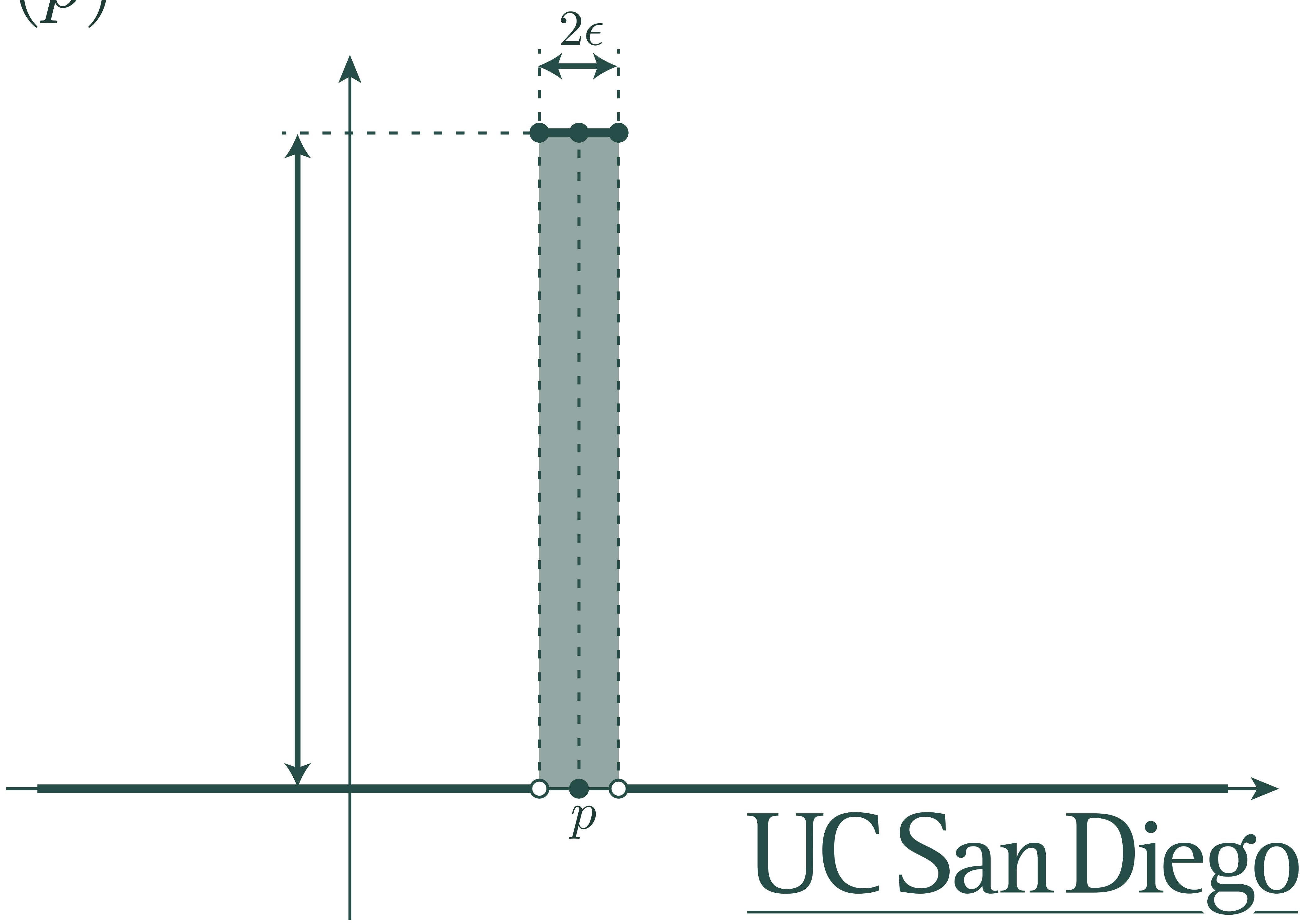
- "Fuzzy" version:

$$\delta_p(\mathbf{x}) \approx \begin{cases} \dfrac{1}{\epsilon^3 |B_1(\mathbf{0})|}, \mathbf{x} \in B_\epsilon(p) \\ 0, \text{otherwise} \end{cases}$$



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Dirac-delta "form"

linear functional on smooth k-forms

- $\Sigma \hookrightarrow M$ is a k-dimensional submanifold
- Represented by a linear functional on smooth k-forms

$$\delta_\Sigma : \Omega^k(M) \to \mathbb{R}$$
$$\omega \mapsto \int_\Sigma \omega$$

- Denoted as $\displaystyle\int_M \omega \wedge \delta_\Sigma = \int_\Sigma \omega$

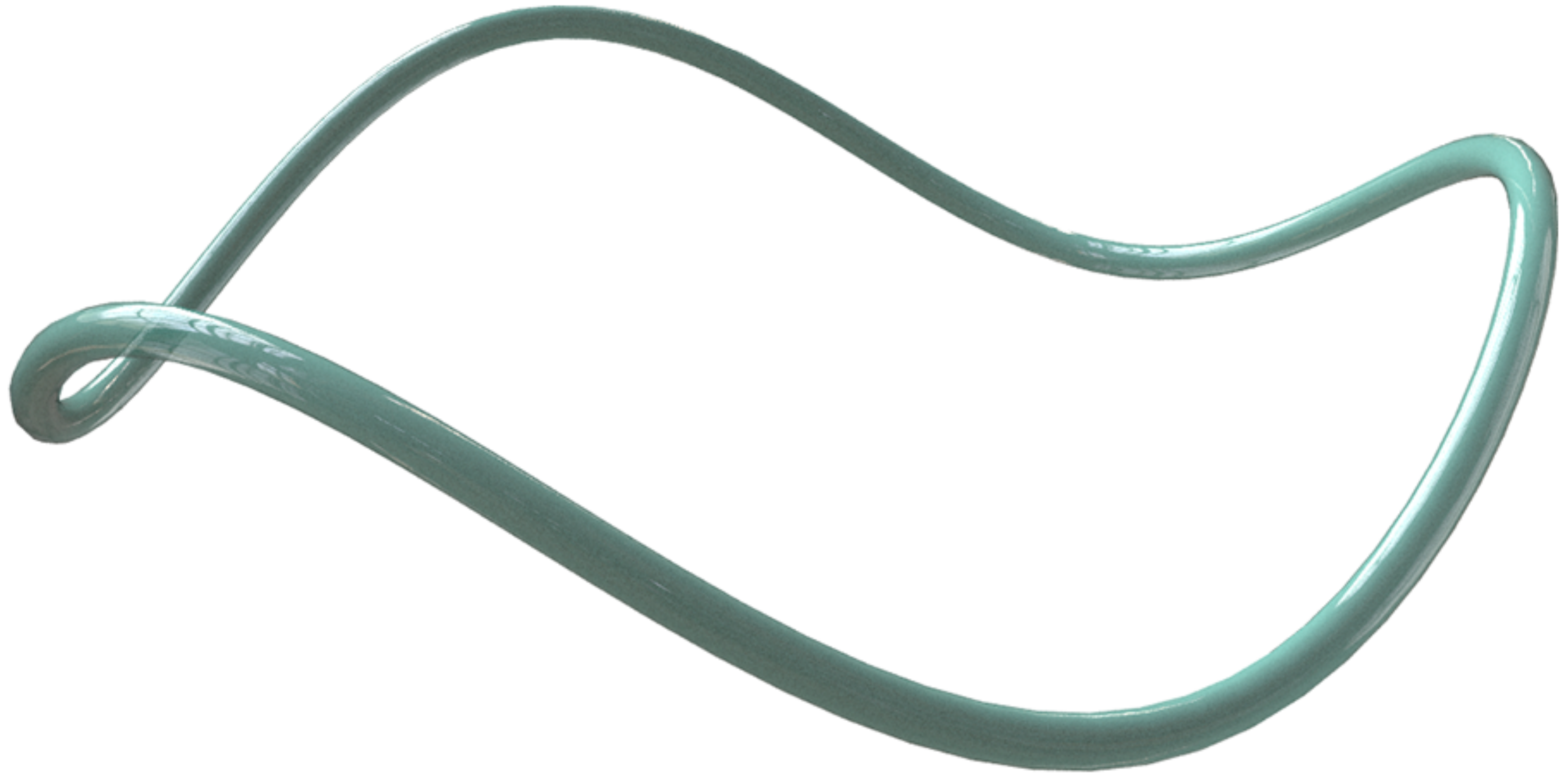
Dirac-delta "form"

linear functional on smooth k-forms

- 1D: curve $\Gamma \hookrightarrow M$
- Line integral $\delta_\Gamma : \eta \in C^\infty\Omega^1(M) \mapsto \int_\Gamma \eta$
- "Fuzzy" version

$$\delta_\Gamma(\mathbf{x}) \approx \begin{cases} \frac{1}{\pi\epsilon^2}\mathbf{t}_\Gamma(\mathbf{x}), \mathbf{x} \in N_\epsilon(\Gamma) \\ 0, \text{otherwise} \end{cases}$$

$$\int_M \mathbf{v} \cdot \delta_\Gamma dV \approx \int_\Gamma \mathbf{v} \cdot \mathbf{t}_\Gamma ds$$



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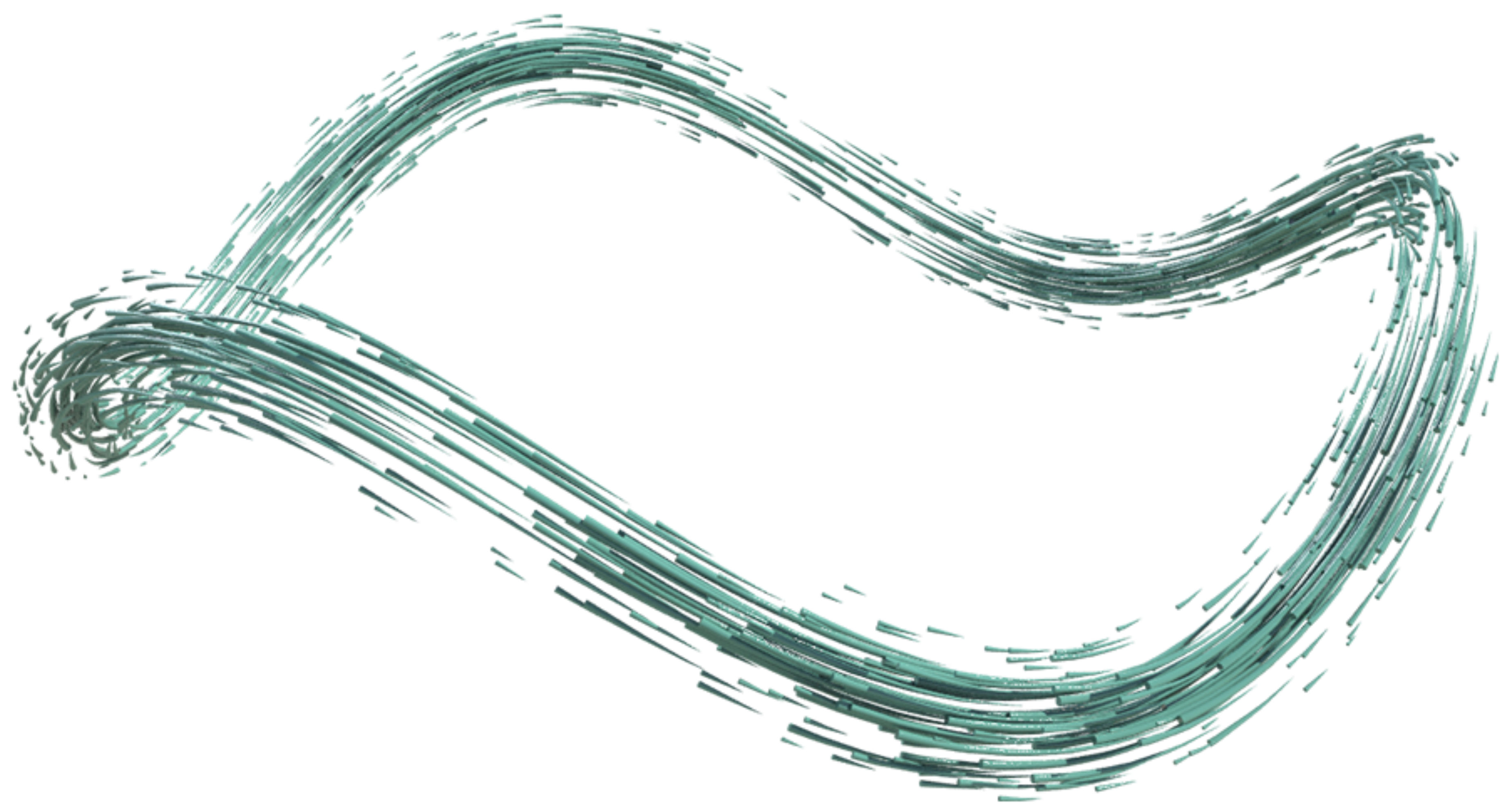

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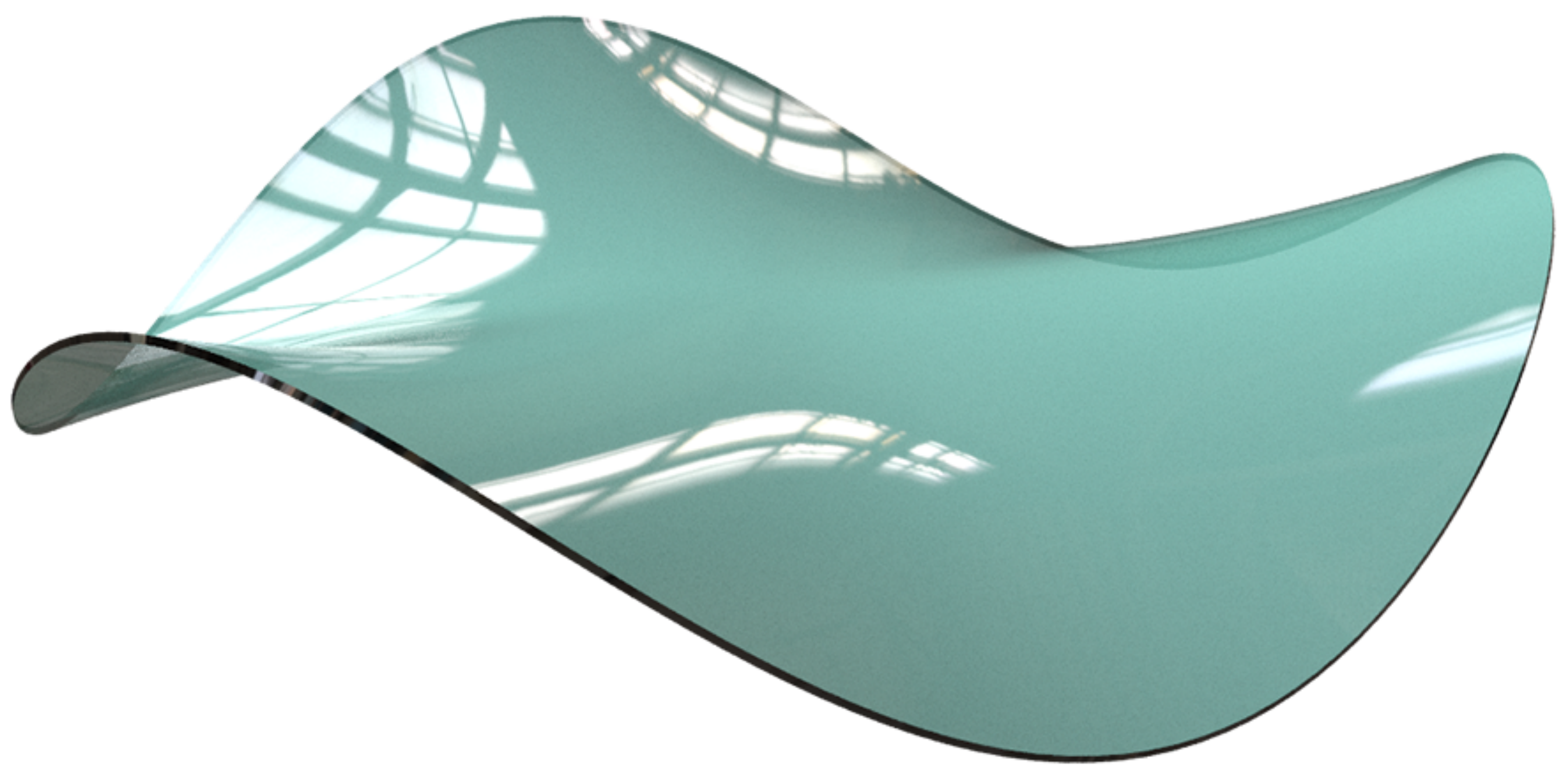
Dirac-delta "form"

linear functional on smooth k-forms



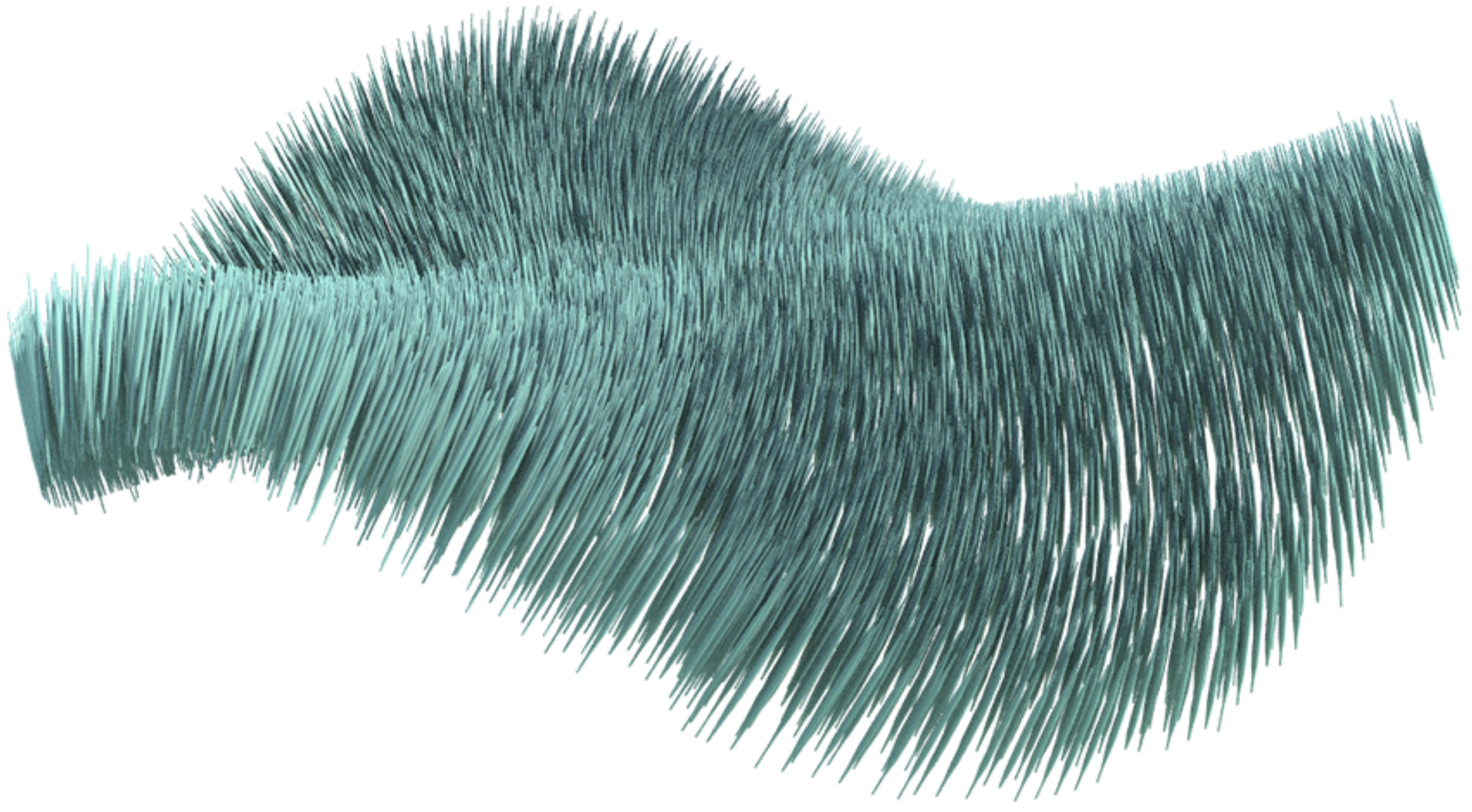
- 2D: surface $\Sigma \hookrightarrow M$
- Flux integral $\delta_\Sigma : \omega \in C^\infty\Omega^2(M) \mapsto \int_\Sigma \omega$
- "Fuzzy" version

$$\delta_\Sigma(\mathbf{x}) \approx \begin{cases} \frac{1}{2\epsilon}\mathbf{n}_\Sigma(\mathbf{x}), & \mathbf{x} \in N_\epsilon(\Sigma) \\ 0, & \text{otherwise} \end{cases}$$

$$\int_M \mathbf{v} \cdot \delta_\Sigma dV \approx \int_\Sigma \mathbf{v} \cdot \mathbf{n}_\Sigma dS$$

Dirac-delta "form"

linear functional on smooth k-forms



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Surface

as taking flux on 2-forms

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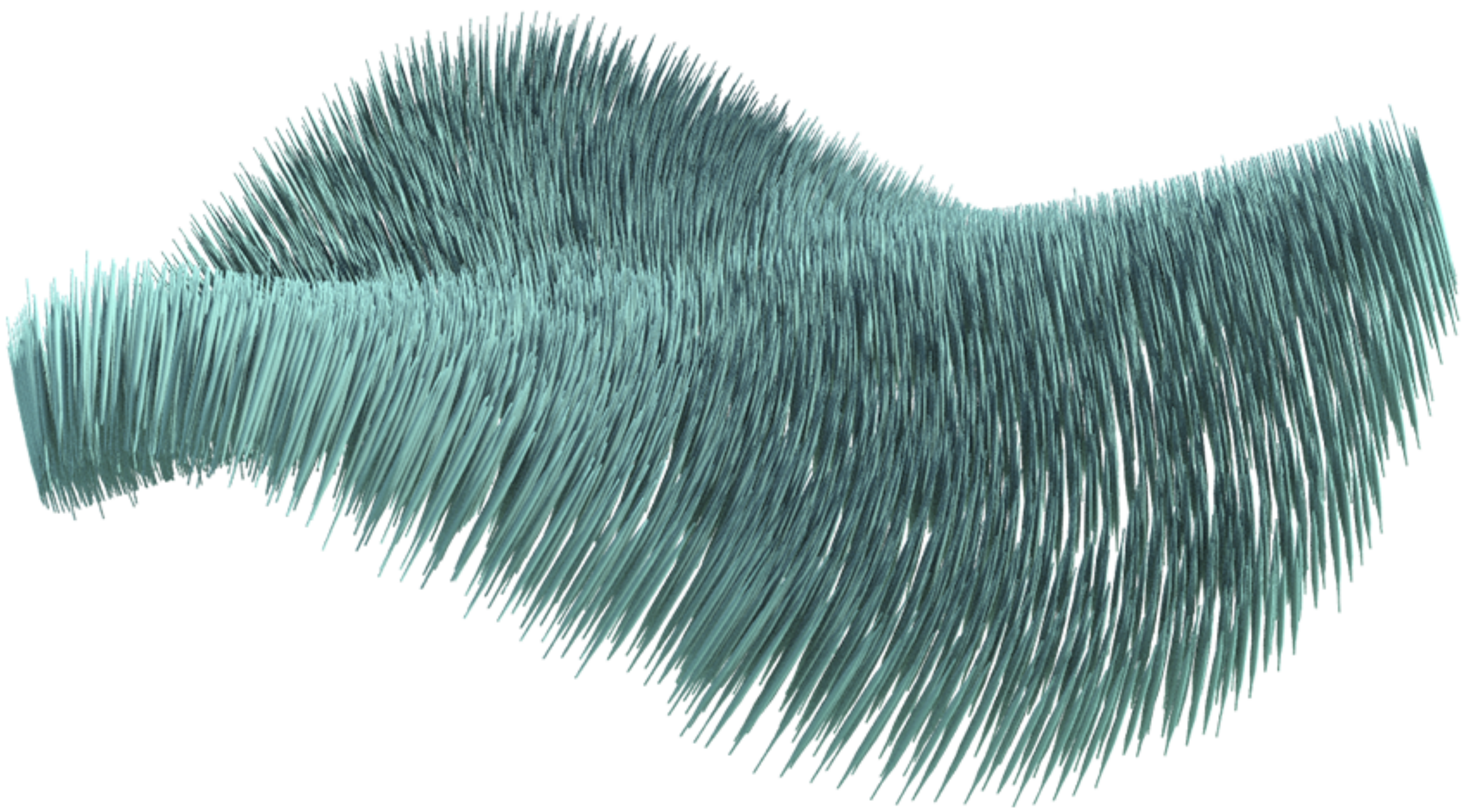

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Dirac-delta "form"

degree

- Linear functional on smooth (compactly-supported) k-forms

$$\int_{\underset{n}{M}} \underset{k}{\omega} \wedge \delta_\Sigma = \int_\Sigma \underset{k}{\omega}$$

Dirac-delta "form"

degree

- Linear functional on smooth (compactly-supported) k-forms

$$\int_{\underset{n}{M}} \underset{k}{\omega} \wedge \underset{n-k}{\delta_\Sigma} = \int_\Sigma \underset{k}{\omega}$$

Dirac-delta “form” to space of current

degree

- Linear functional on smooth (compactly-supported) k-forms

$$\int_{\underset{n}{M}} \underset{k}{\omega} \wedge \underset{n-k}{\delta_\Sigma} = \int_\Sigma \underset{k}{\omega}$$

- Denoted all linear functional on smooth k-forms by

$$\mathcal{D}\Omega^{n-k}(M) = (\Omega^k(M))^* = (C^\infty(M))^* \otimes \{dx_I : |I| = n-k\}$$

- For easy distinction, denote smooth k-forms as $C^\infty\Omega^k(M)$

Currents

properties

- Linear combination

$$\alpha_1 \delta_{\Sigma_1} + \cdots + \alpha_m \delta_{\Sigma_m} : \omega \in C^\infty \Omega^1(M) \mapsto \alpha_1 \int_{\Sigma_1} \omega + \cdots + \alpha_m \int_{\Sigma_m} \omega$$

  - Superposition of submanifolds $\alpha_1 \Sigma_1 + \cdots + \alpha_m \Sigma_m$

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Surface $\Sigma \hookrightarrow M$

Dirac-delta 1-form $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$

Currents

Stokes' theorem

- Stokes' Theorem $\int_{\partial N} \nu = \int_N d\nu$

$$\delta_{\partial N}[\nu] = \delta_N \circ d[\nu]$$

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Currents

weak derivatives

$$\delta_{\partial N}[\nu] = \delta_N \circ d[\nu]$$

- $g = Df$ in the weak sense if $\forall \varphi \in C_0^\infty, \int \varphi g = \oint \varphi f - \int (D\varphi) f$
  - As operation on φ, weak derivative $\langle \varphi, Df \rangle = \langle d\varphi, f \rangle + \text{boundary term}$

Currents

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- $d\eta$ is the weak derivative of $\eta \in \mathcal{D}\Omega^{n-k}(M)$ if

$$\forall \omega \in C^\infty \Omega^{k-1}(M), (-1)^{k-1} \int_M \omega \wedge d\eta = \oint_{\partial M} \omega \wedge \eta - \int_M d\omega \wedge \eta$$

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Currents

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- For Dirac-delta form $\eta = \delta_\Sigma$ (k=2),

$$\begin{aligned}(-1)\int_M \omega \wedge d\delta_\Sigma &= \oint_{\partial M} \omega \wedge \delta_\Sigma - \int_M d\omega \wedge \delta_\Sigma \\ &= -\int_M d\omega \wedge \delta_\Sigma \\ &= -\oint_{\partial\Sigma} \omega = -\int_M \omega \wedge \delta_{\partial\Sigma}\end{aligned}$$

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Currents

weak derivatives

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Surface $\Sigma \hookrightarrow M$

Dirac-delta 1-form $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$

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Surface $\Sigma \hookrightarrow M$
Boundary $\partial\Sigma = \Gamma$

Dirac-delta 1-form $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$
Weak derivative $d\delta_\Sigma = \delta_\Gamma$

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Current mass norm

Operator norm w.r.t. sup-norm on smooth forms

- Operator norm for (n-k)-current $\eta : C^\infty \Omega^k(M) \to \mathbb{R}$

$$\|\eta\|_{\mathrm{mass}} = \sup_{\omega \in C^\infty \Omega^k(M), \|\omega\|_{L^\infty} \leq 1} |\eta[\omega]|$$

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- Sup-norm for k-smooth form $\omega \in C^\infty\Omega^k(M)$

$$\|\omega\|_{L^\infty} = \sup_{p\in M} |\omega|_p$$

- Pointwise Euclidean ℓ^2 norm

$$|\omega| : M \to \mathbb{R}^{\geq 0}$$

$$p \mapsto \sqrt{\star(\omega \wedge \star\omega)_p}$$

$$\omega = \sum_I f_I dx_I, |\omega|(p) = \sqrt{\sum_I f_I(p)^2}$$

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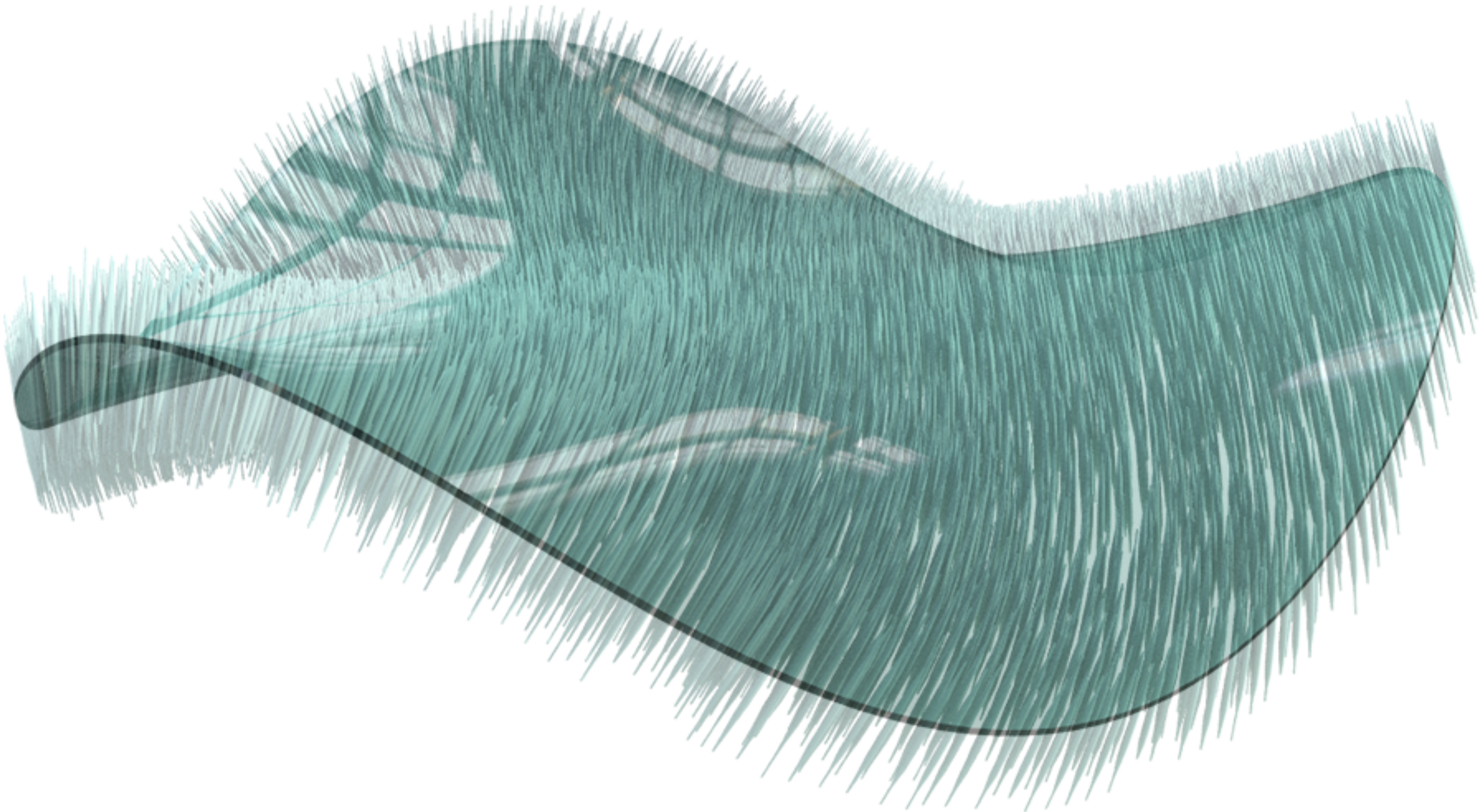
Current mass norm

equivalent to surface area on Dirac-delta 1-forms

- Dirac-delta form for surfaces $\delta_\Sigma \approx \frac{1}{\epsilon}\mathbf{n}_\Sigma$



$$\begin{aligned} \|\delta_\Sigma\|_{\text{mass}} &= \sup_{\substack{\omega \in C^\infty\Omega^k(M) \\ \|\omega\|_{L^\infty} \leq 1}} \left| \int_M \omega \wedge \delta_\Sigma \right| \\ &= \sup_{\substack{\mathbf{v} \in C^\infty(M \to \mathbb{R}^3) \\ |\mathbf{v}| \leq 1}} \int_\Sigma \mathbf{v} \cdot \mathbf{n}_\Sigma dS \\ &= \int_\Sigma \mathbf{n}_\Sigma \cdot \mathbf{n}_\Sigma dS = \int_\Sigma 1 dS = \text{Area}(\Sigma) \end{aligned}$$

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Surface $\Sigma \hookrightarrow M$

Boundary $\partial\Sigma = \Gamma$

Dirac-delta 1-form $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$

Weak derivative $d\delta_\Sigma = \delta_\Gamma$

Surface $\Sigma \hookrightarrow M$

Boundary $\partial\Sigma = \Gamma$

Surface area $\mathrm{Area}(\Sigma)$

Dirac-delta 1-form $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$

Weak derivative $d\delta_\Sigma = \delta_\Gamma$

Mass norm $\|\delta_\Sigma\|_{\mathrm{mass}}$

Plateau problem

original problem

- Variable $\Sigma \hookrightarrow M$
- Constraint $\partial\Sigma = \Gamma$
- Objective $\mathrm{Area}(\Sigma)$

Plateau problem

written in terms of current

- Variable $\Sigma \hookrightarrow M$
- Constraint $\partial\Sigma = \Gamma$
- Objective $\text{Area}(\Sigma)$

- represented by $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$
- equivalently, $d\delta_\Sigma = \delta_\Gamma$
- equivalently, $\|\delta_\Sigma\|_{\text{mass}}$

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Plateau problem, relaxed

written in terms of current

- Variable $\Sigma \hookrightarrow M$
- Constraint $\partial\Sigma = \Gamma$
- Objective $\text{Area}(\Sigma)$

- represented by $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$
- equivalently, $d\delta_\Sigma = \delta_\Gamma$
- equivalently, $\|\delta_\Sigma\|_{\text{mass}}$

- Variable $\eta \in \mathcal{D}\Omega^1(M)$
- Constraint $d\eta = \delta_\Gamma$
- Objective $\|\eta\|_{\text{mass}}$

Plateau problem, relaxed

written in terms of current

- Variable $\Sigma \hookrightarrow M$
- Constraint $\partial\Sigma = \Gamma$
- Objective $\text{Area}(\Sigma)$

- represented by $\delta_\Sigma \in \mathcal{D}\Omega^1(M)$
- equivalently, $d\delta_\Sigma = \delta_\Gamma$
- equivalently, $\|\delta_\Sigma\|_{\text{mass}}$

- Variable $\eta \in \mathcal{D}\Omega^1(M)$
- Constraint $d\eta = \delta_\Gamma$
- Objective $\|\eta\|_{\text{mass}}$

$$\min_{\partial\Sigma=\Gamma} \|\delta_\Sigma\|_{\text{mass}} = \min_{d\eta=\delta_\Gamma} \|\eta\|_{\text{mass}}$$

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“Area functional is no longer nonconvex.”

- Σ_1, Σ_2 are symmetric minima
- $\|\eta\|_{\text{mass}}$ is constant on segment $\{\theta\delta_{\Sigma_1} + (1-\theta)\delta_{\Sigma_2} : 0 \leq \theta \leq 1\}$

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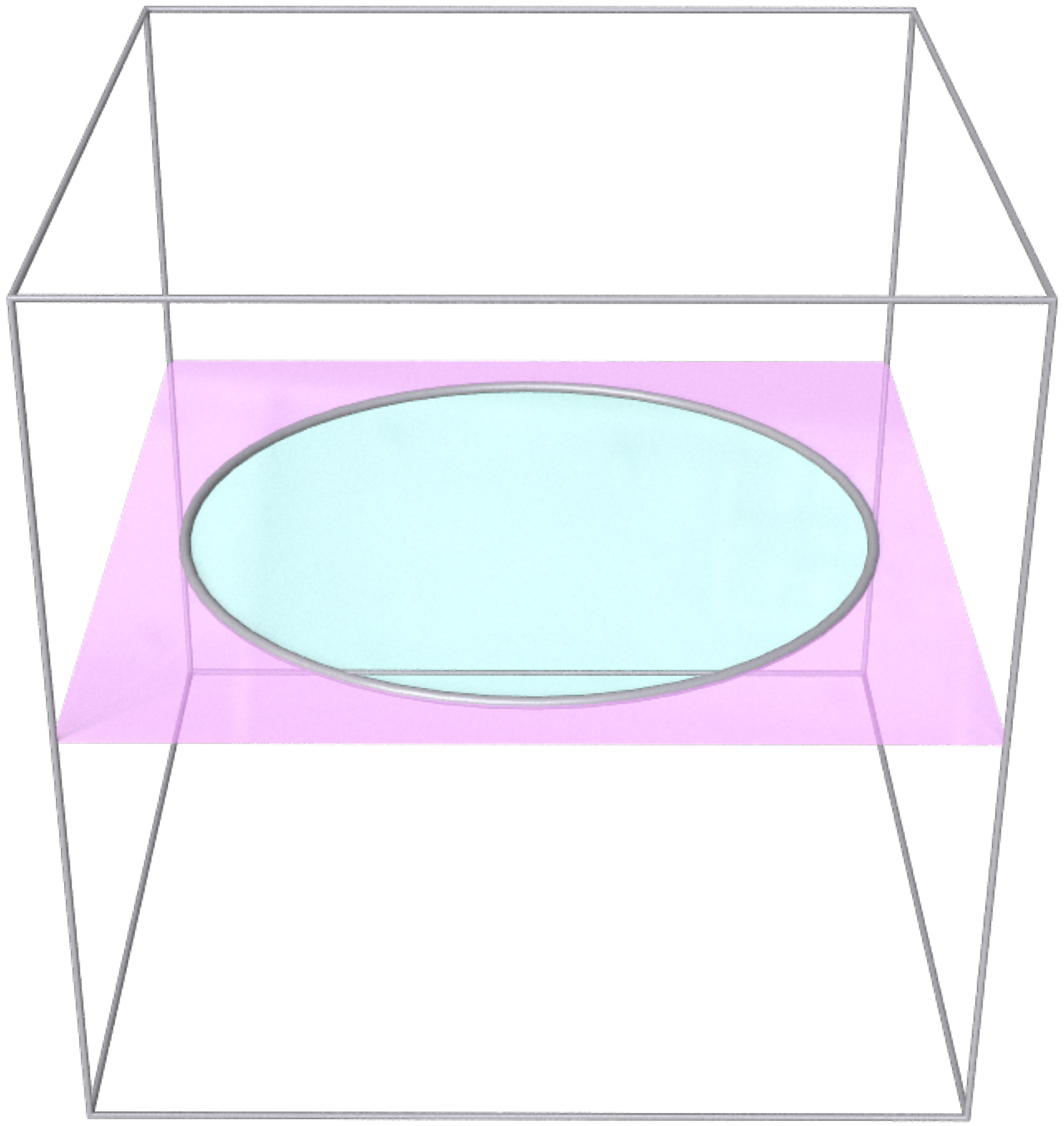
Plateau problem on 3-torus

periodic boundary artifact



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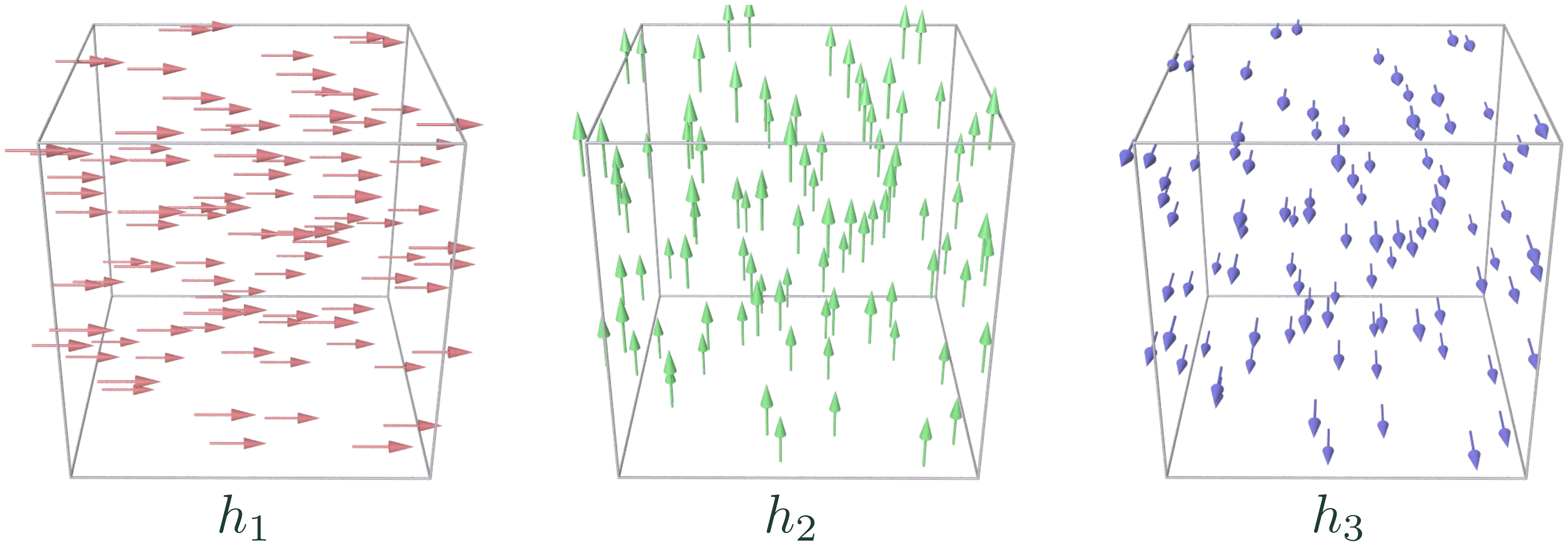
Admissible set

Cohomology matters

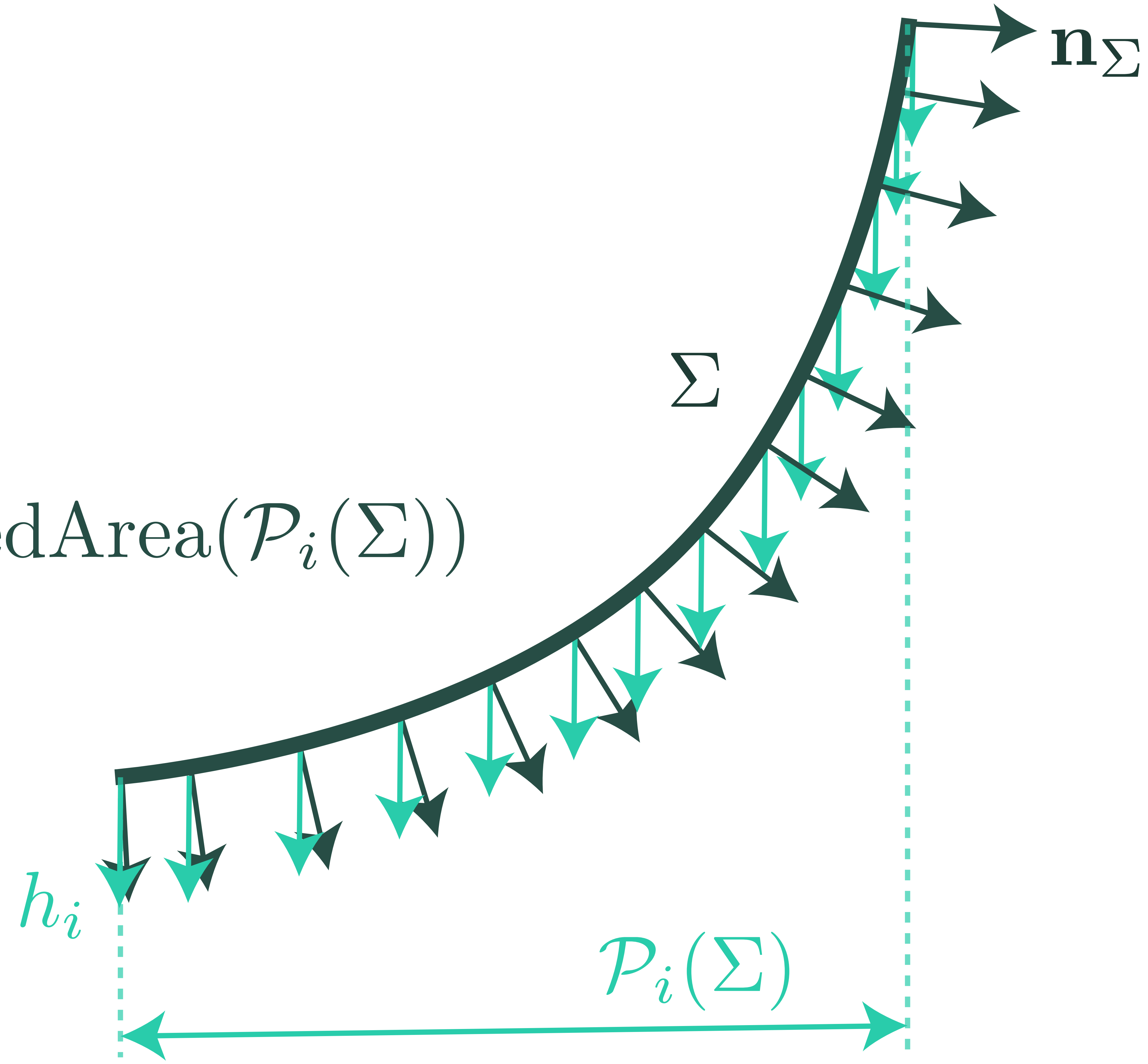
- Nontrivial cohomology $\mathcal{H}^1(\mathbb{T}^3) = \ker(d^1)/\mathrm{im}(d^0) \neq 0$
- Nontrivial harmonic forms (closed but not exact)



Admissible set

Cohomology as projected area

$$\int h_i \wedge \delta_\Sigma = \int_\Sigma \mathbf{e}_i \cdot \mathbf{n}_\Sigma dS$$
$$= \int_{\mathcal{P}_i(\Sigma)} 1 dS = \text{SignedArea}(\mathcal{P}_i(\Sigma))$$



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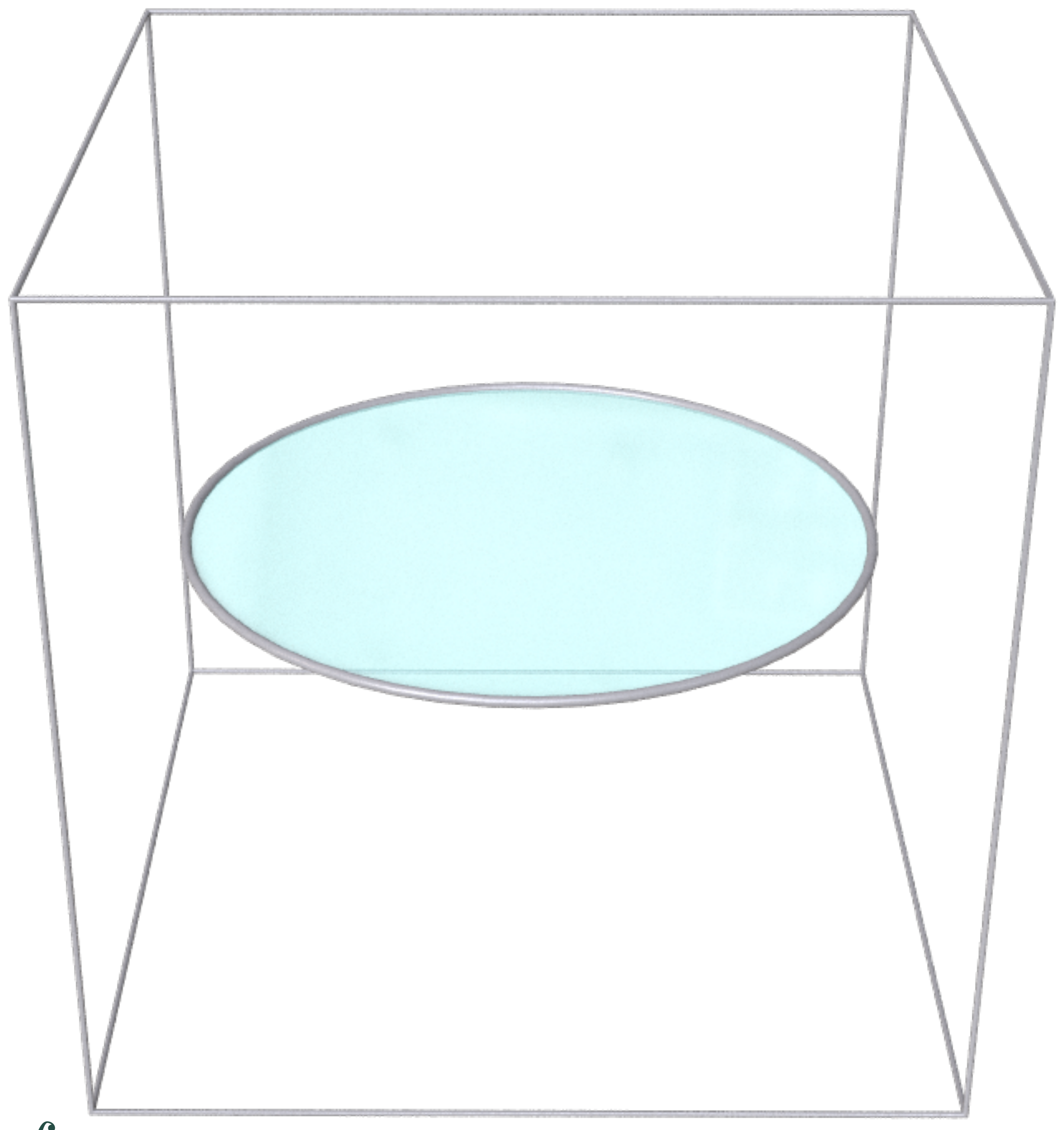
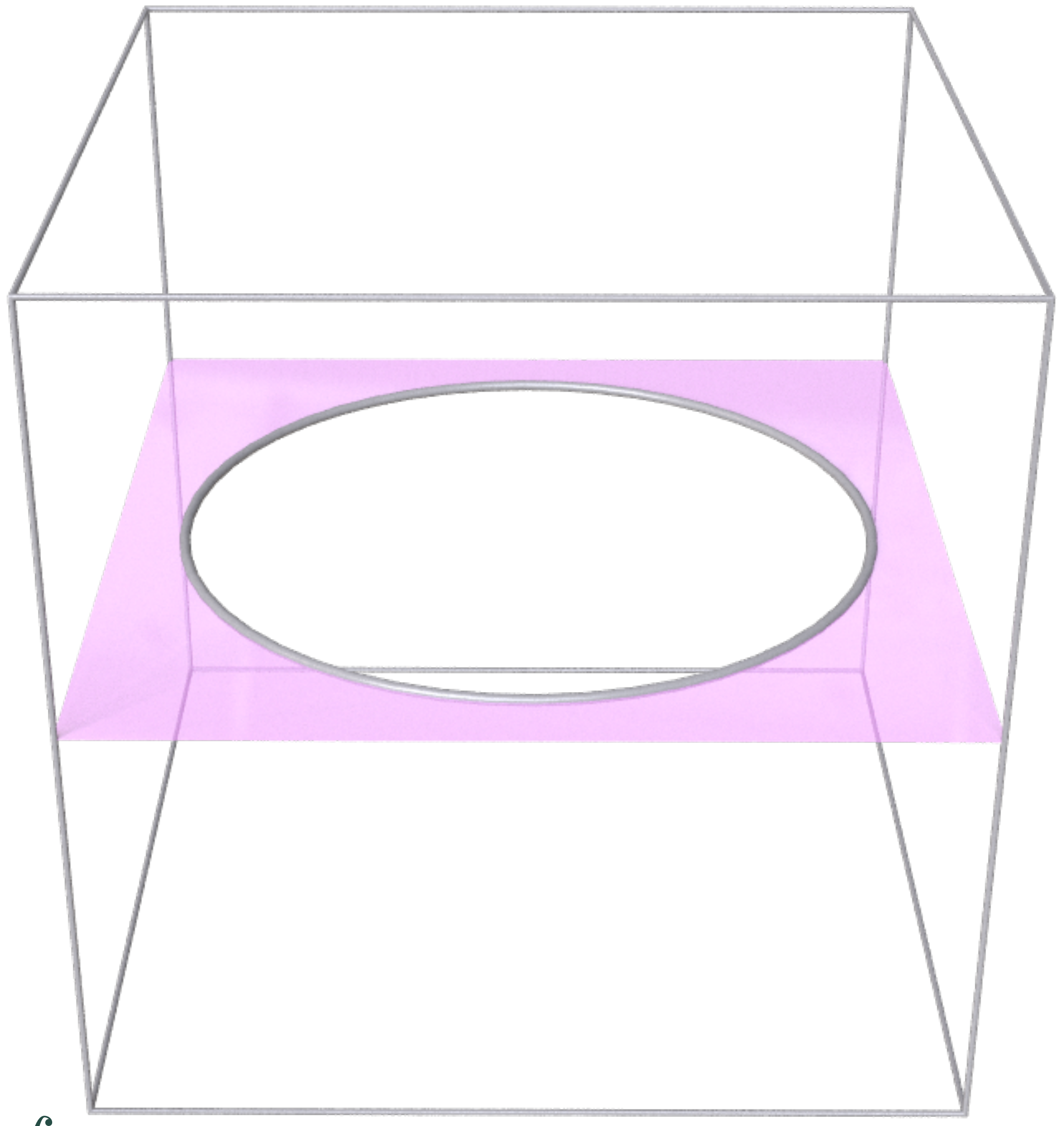
Admissible set

examples

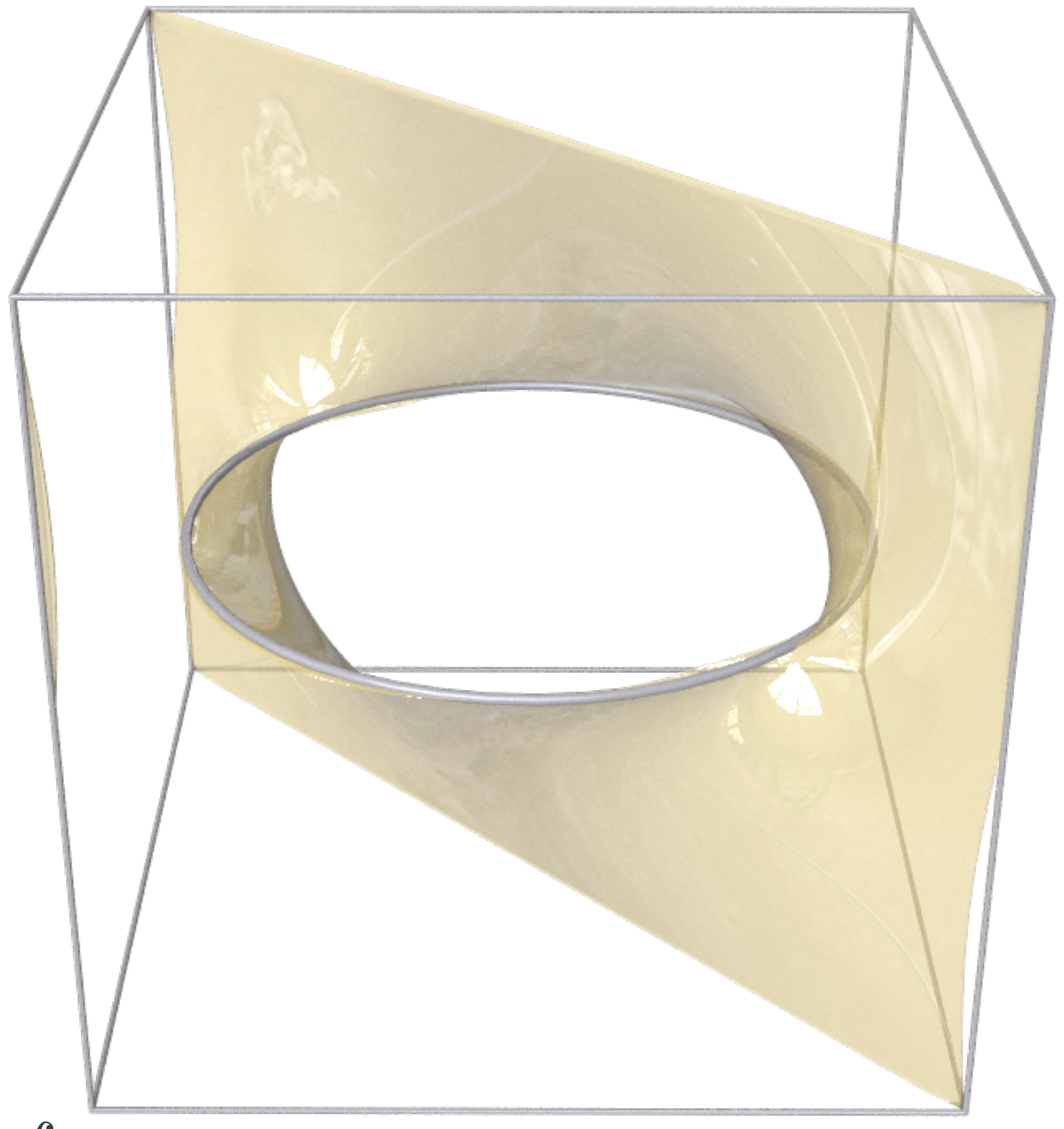
$$\int h_i \wedge \delta_\Sigma = \int_\Sigma \mathbf{e}_i \cdot \mathbf{n}_\Sigma dS$$
$$= \int_{\mathcal{P}_i(\Sigma)} 1 dS = \mathrm{SignedArea}(\mathcal{P}_i(\Sigma))$$



$$\int h_1 \wedge \eta = 0$$
$$\int h_2 \wedge \eta = \mathrm{Area}(\mathbb{D})$$
$$\int h_3 \wedge \eta = 0$$



$$\int h_1 \wedge \eta = 0$$
$$\int h_2 \wedge \eta = 1 - \mathrm{Area}(\mathbb{D})$$
$$\int h_3 \wedge \eta = 0$$



$$\int h_1 \wedge \eta = -1$$
$$\int h_2 \wedge \eta = 1 - \mathrm{Area}(\mathbb{D})$$
$$\int h_3 \wedge \eta = 1$$

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Admissible set

Cohomology constraint

- Adding Cohomology constraint

$$\int h_i \wedge \eta = \psi_i, i = 1, 2, 3$$

- Admissible set

$$\mathcal{A} = \left\{ \eta \in \mathcal{D}\Omega^1(M) \middle| d\eta = \delta_\Gamma, \int h_i \wedge \eta = \psi_i, i = 1, 2, 3 \right\}$$

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Admissible set

Cohomology constraint

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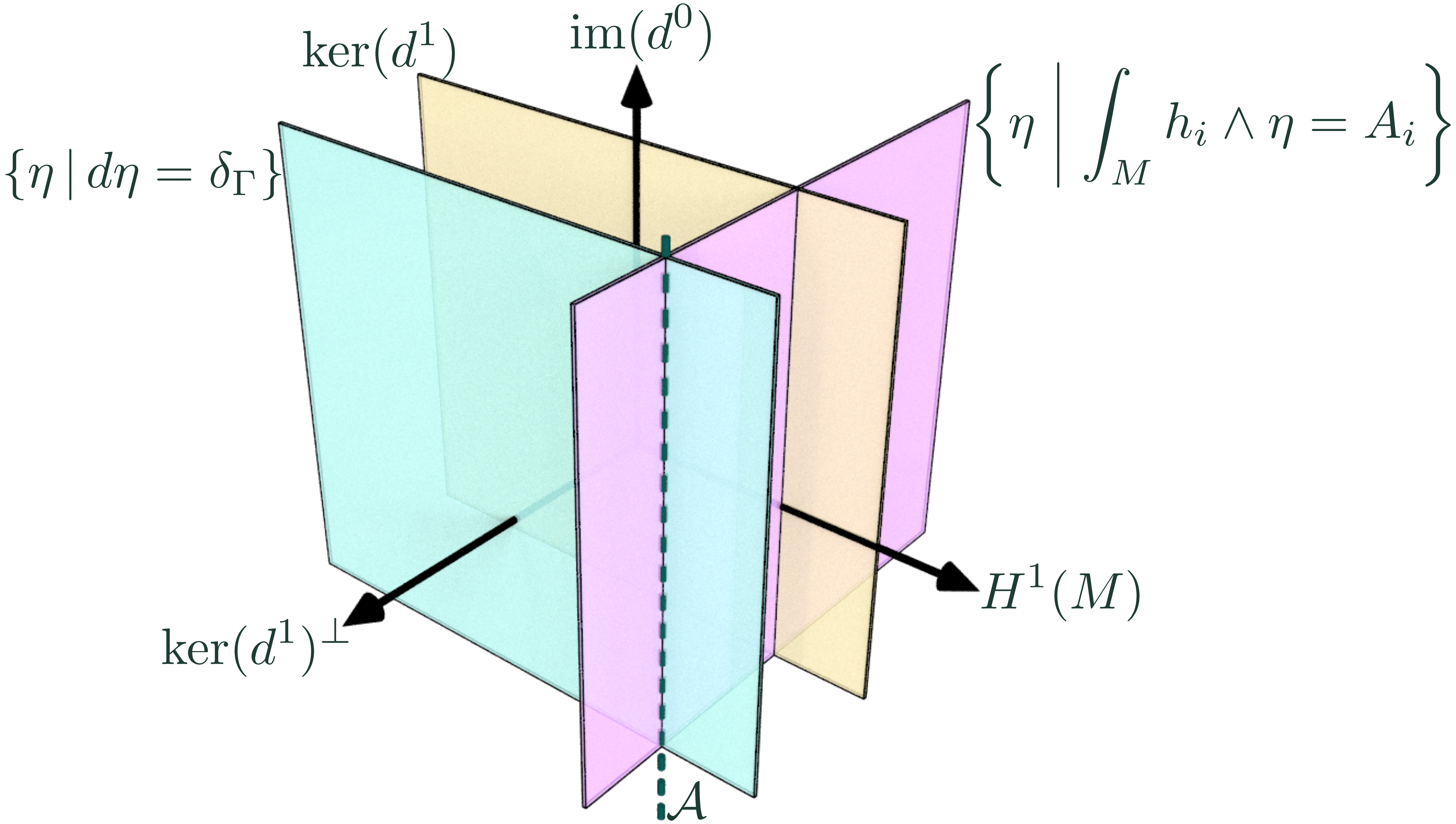

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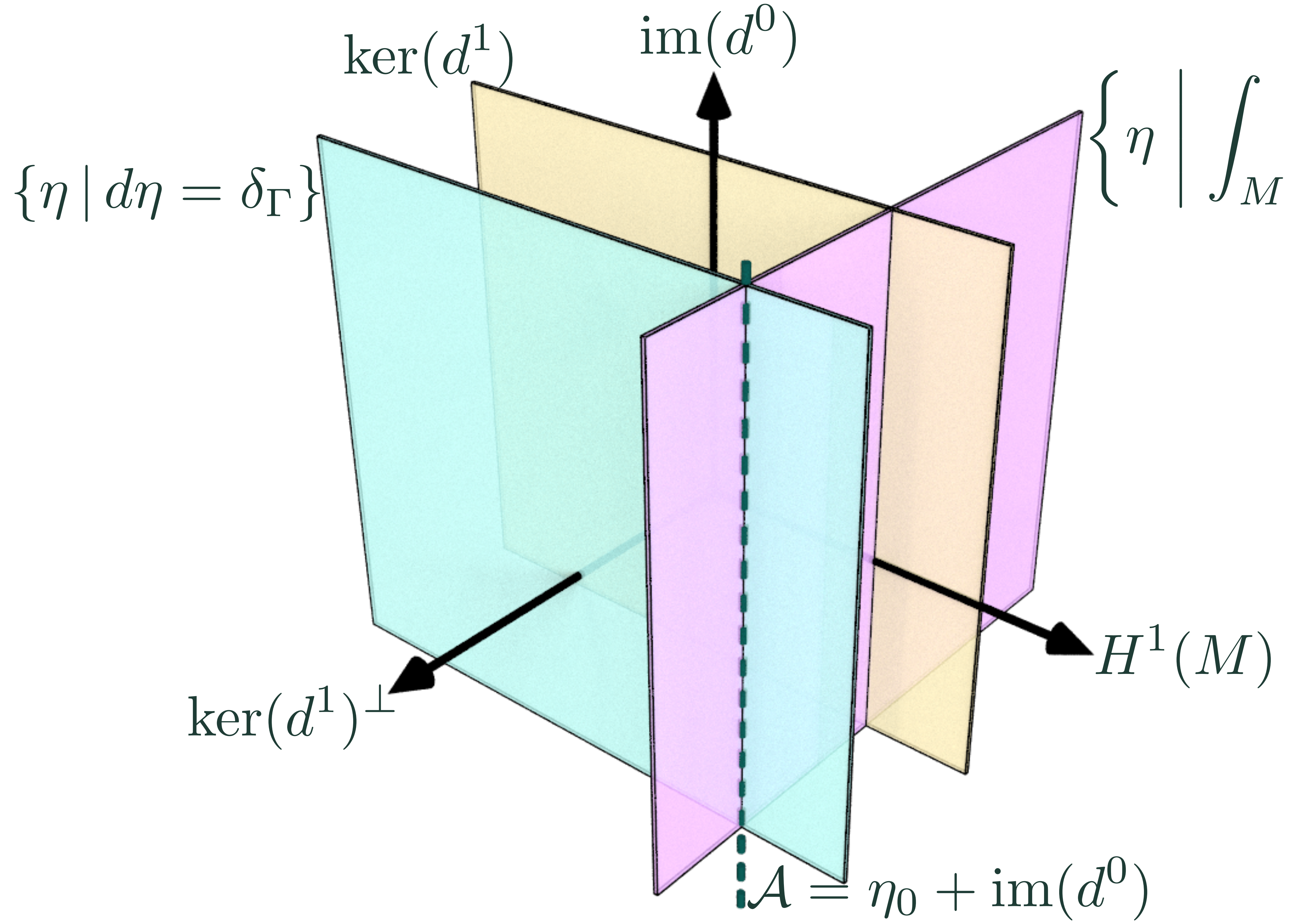
Admissible set

Cohomology constraint

$$\mathcal{A} = \left\{ \eta \in \mathcal{D}\Omega^1(M) \middle| d\eta = \delta_\Gamma, \int h_i \wedge \eta = \psi_i, i = 1, 2, 3 \right\} = \eta_0 + \mathrm{im}\left(d^0\right)$$



Fast ADMM

Equivalent formulation

- Initial guess $\eta_0 \in \mathcal{A}$ by Biot Savart

$$\begin{aligned} \text{minimize} \quad & \|\eta\|_{\text{mass}} \\ \text{s.t.} \quad & \eta \in \eta_0 + \operatorname{im}\left(d^0\right) \end{aligned}$$

Fast ADMM

Equivalent formulation

- Initial guess $\eta_0 \in \mathcal{A}$ by Biot Savart

$$\begin{aligned} \text{minimize} \quad & \|\eta\|_{\text{mass}} \\ \text{s.t.} \quad & \eta \in \eta_0 + \operatorname{im}\left(d^0\right) \end{aligned}$$

- Equivalently problem:

$$\begin{aligned} \text{variable} \quad & \varphi \in \Omega^0(M), \eta \in \Omega^1(M) \\ \text{minimize} \quad & \|\eta\|_{\text{mass}} \\ \text{s.t.} \quad & \eta - \eta_0 = d\varphi \end{aligned}$$

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Fast ADMM

global (Poisson) and local (Shrink) solve

- Global

$$\min_{\varphi} \langle \lambda, d\varphi \rangle + \frac{\tau}{2} \| d\varphi - \eta + \eta_0 \|^2_{L^2}$$

- $\tau \Delta \varphi = \eta - \eta_0 + \delta \lambda$
- Poisson equation on torus
- Spectral method

- Local

$$\min_{\eta} \|\eta\|_{\mathrm{mass}} - \langle \lambda, \eta \rangle + \frac{\tau}{2} \| d\varphi - \eta + \eta_0 \|^2_{L^2}$$

- $\forall p \in M, (\tau(d\varphi - \eta + \eta_0) + \lambda)_p \in \partial |\eta_p|$
- L1 pointwise optimality
- Shrinkage operator

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Results

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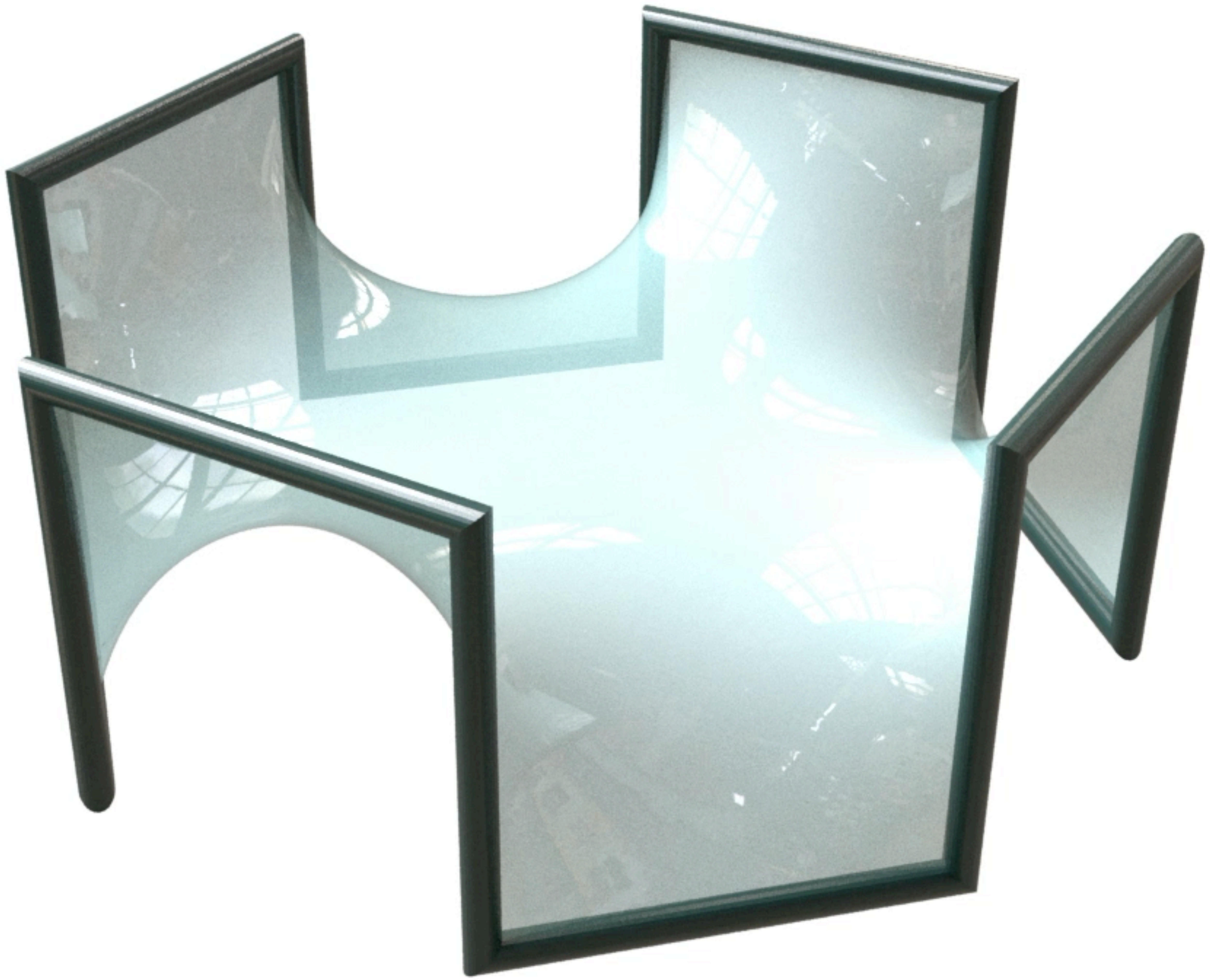
Converting current to level set

- Optimal solution $\eta^* = \delta_{\Sigma^*}$
  singularity at Σ^* in normal direction
- $u^* = d^+\eta^*$ jump at Σ^* in normal direction
- Take level set $\Sigma^* = \{x \in M | u^*(x) = 0\}$



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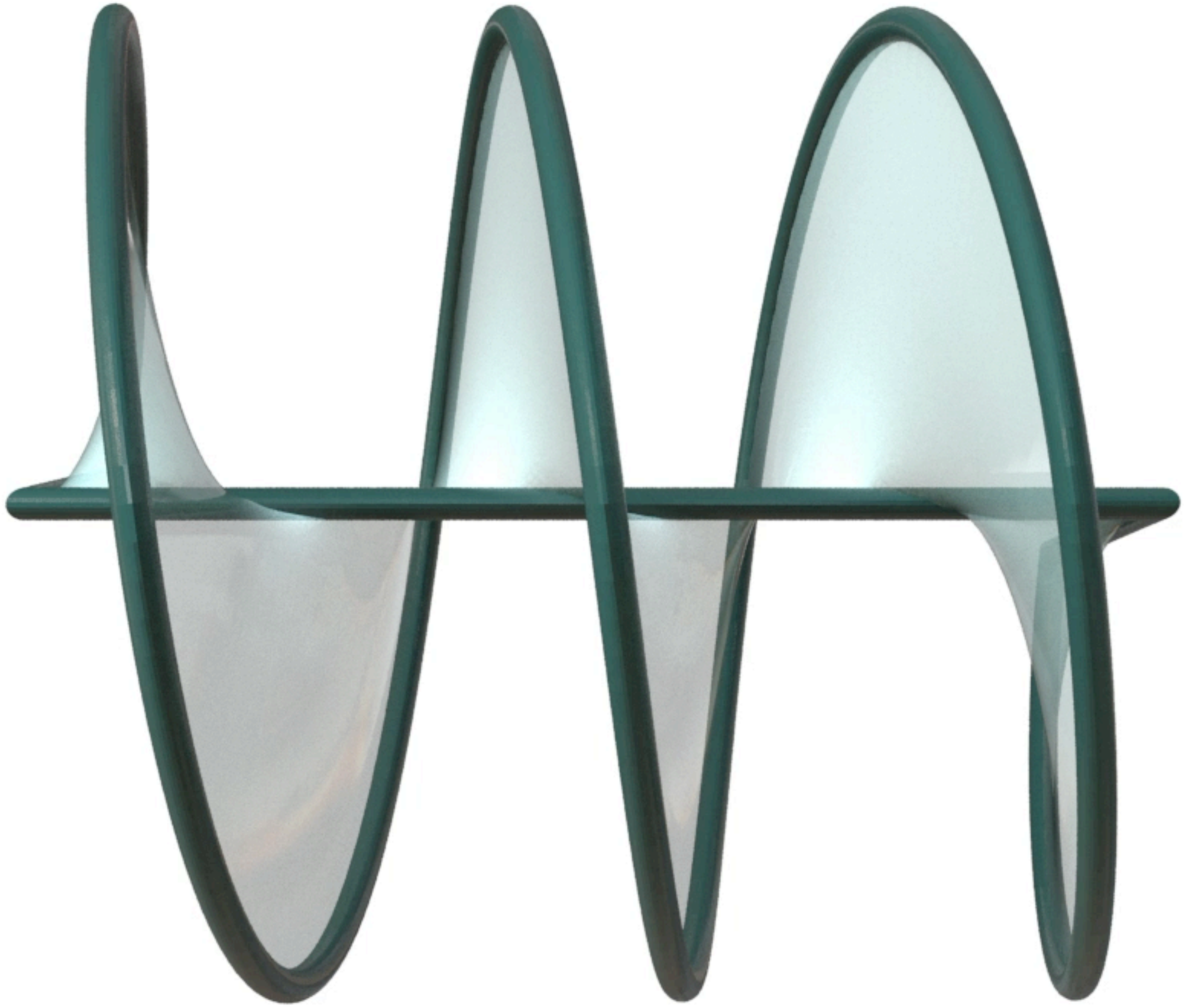
  singularity at Σ^* in normal direction
- $u^* = d^+\eta^*$ jump at Σ^* in normal direction
- Take level set $\Sigma^* = \{x \in M | u^*(x) = 0\}$
- $d^+\eta^* = \underset{u \in \mathcal{D}\Omega^0(M):du=\eta^*}{\operatorname{argmin}} \|u\|_{L^2}$
- $d^+\eta^* = \Delta^{-1}\delta\eta^*$ single Poisson solve



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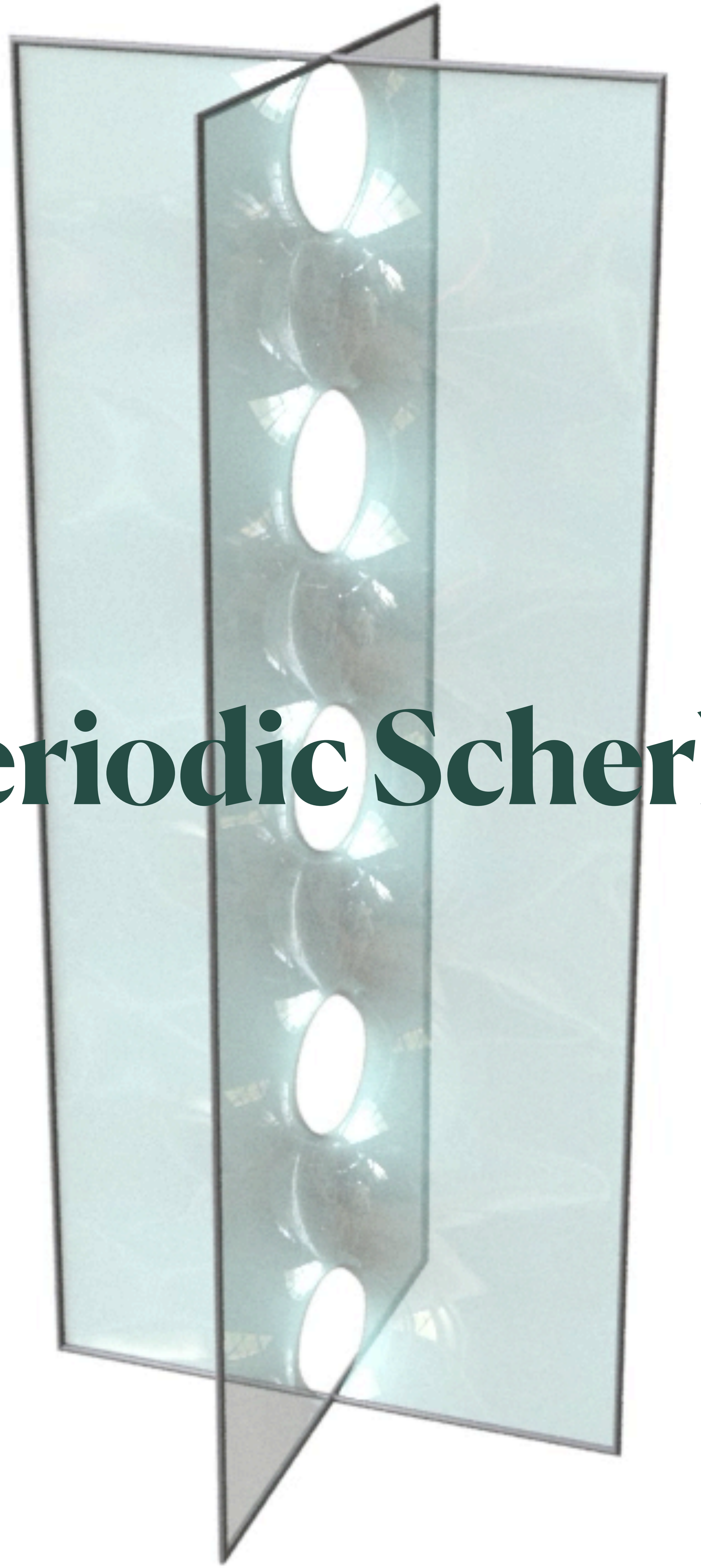

Singly periodic Scherk surface

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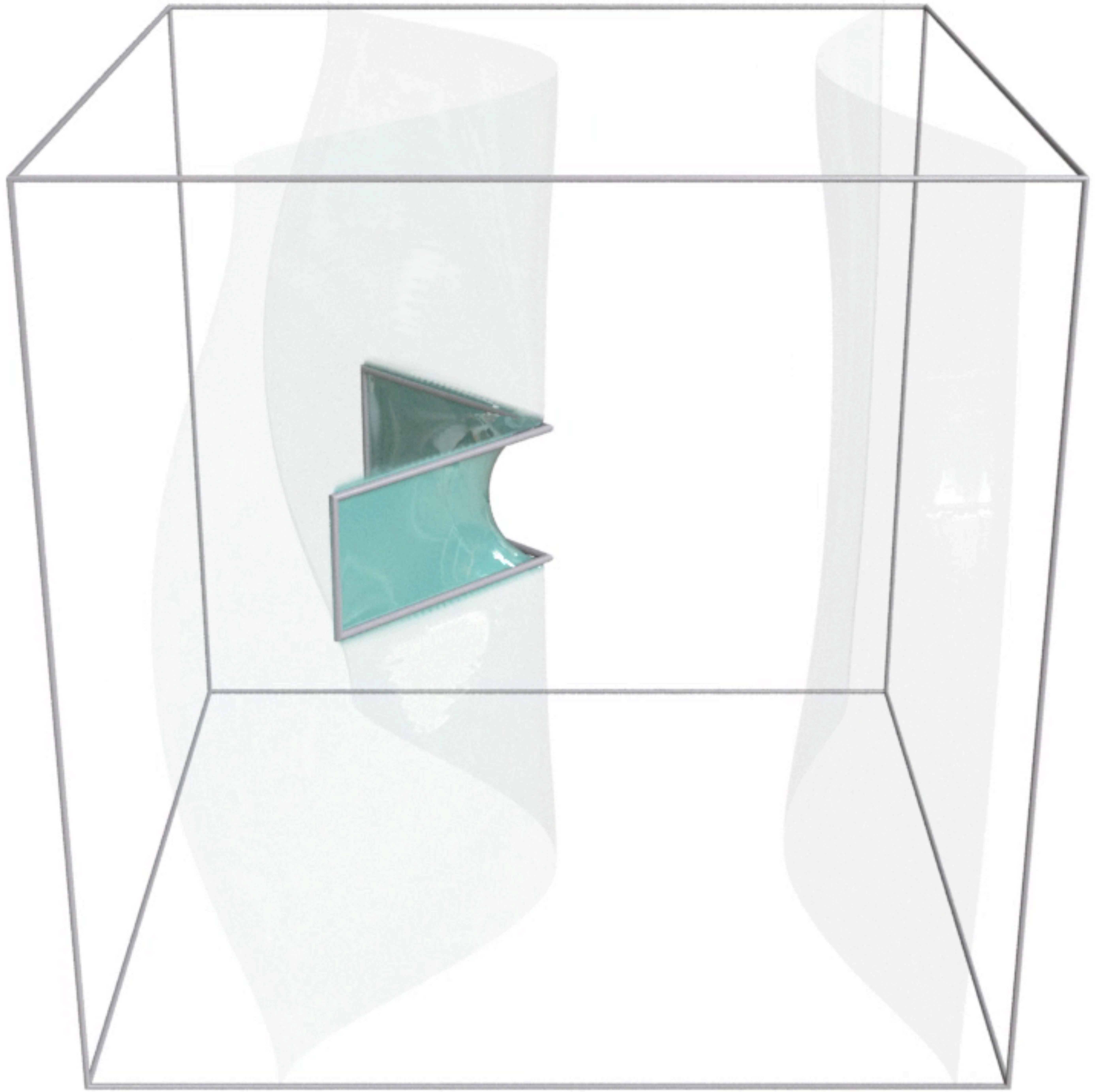
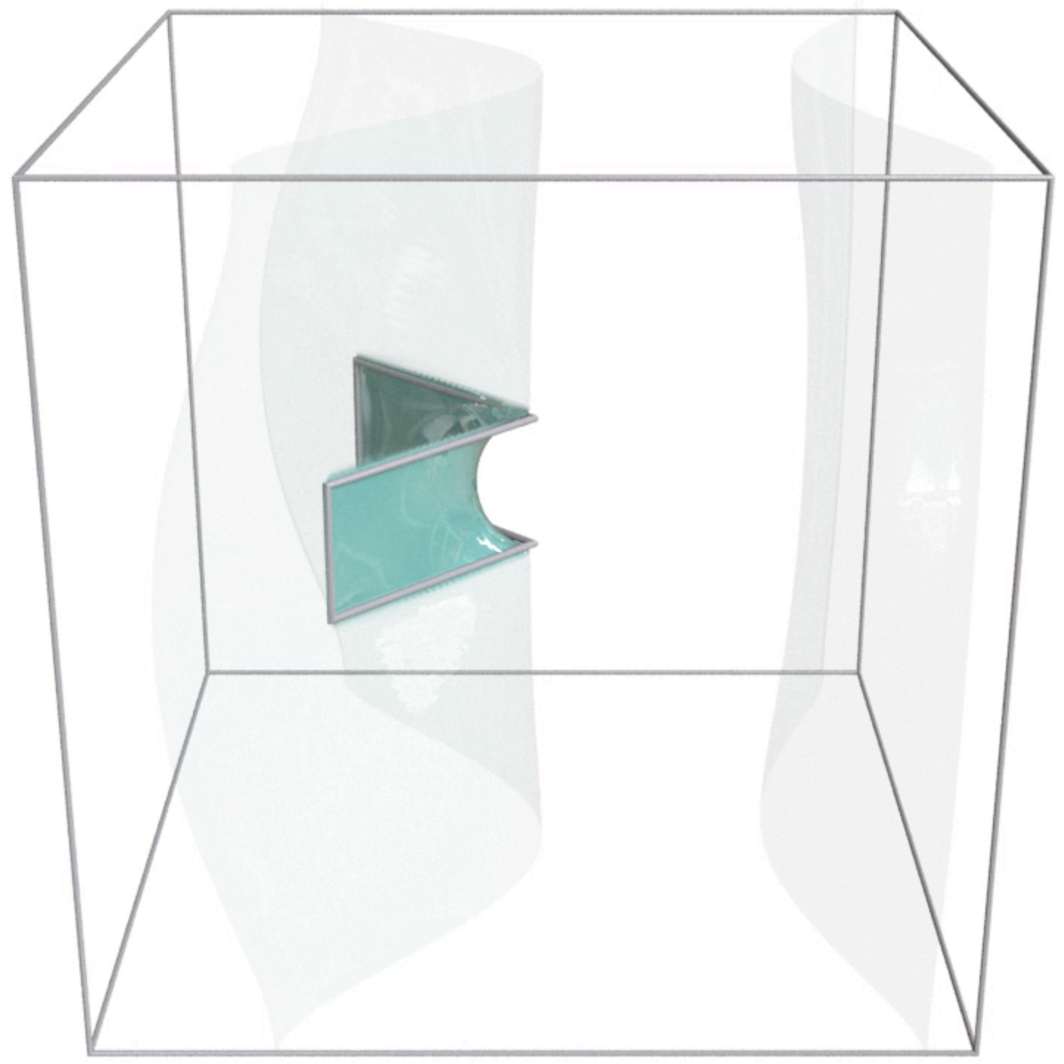
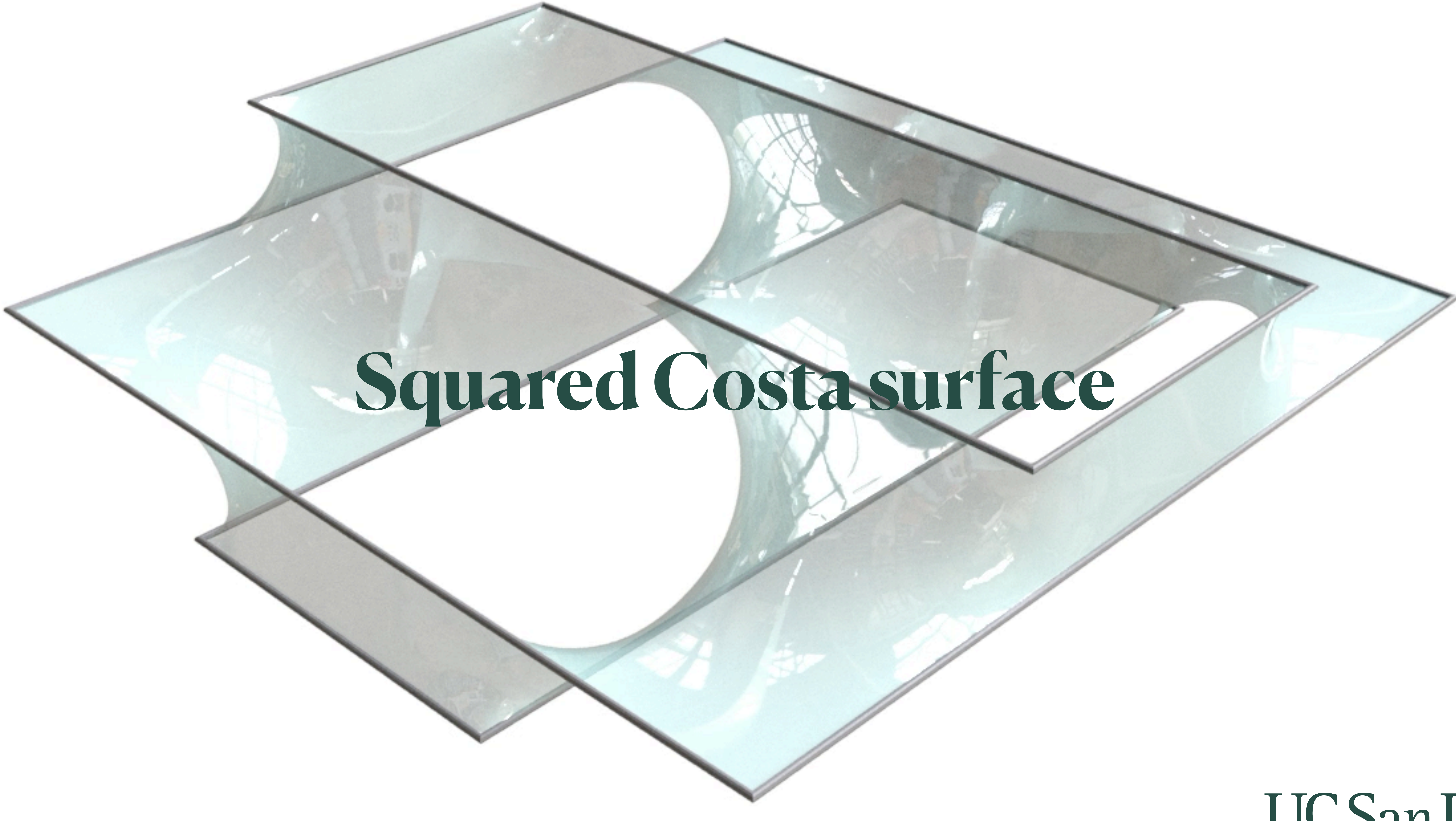




Squared Costa surface



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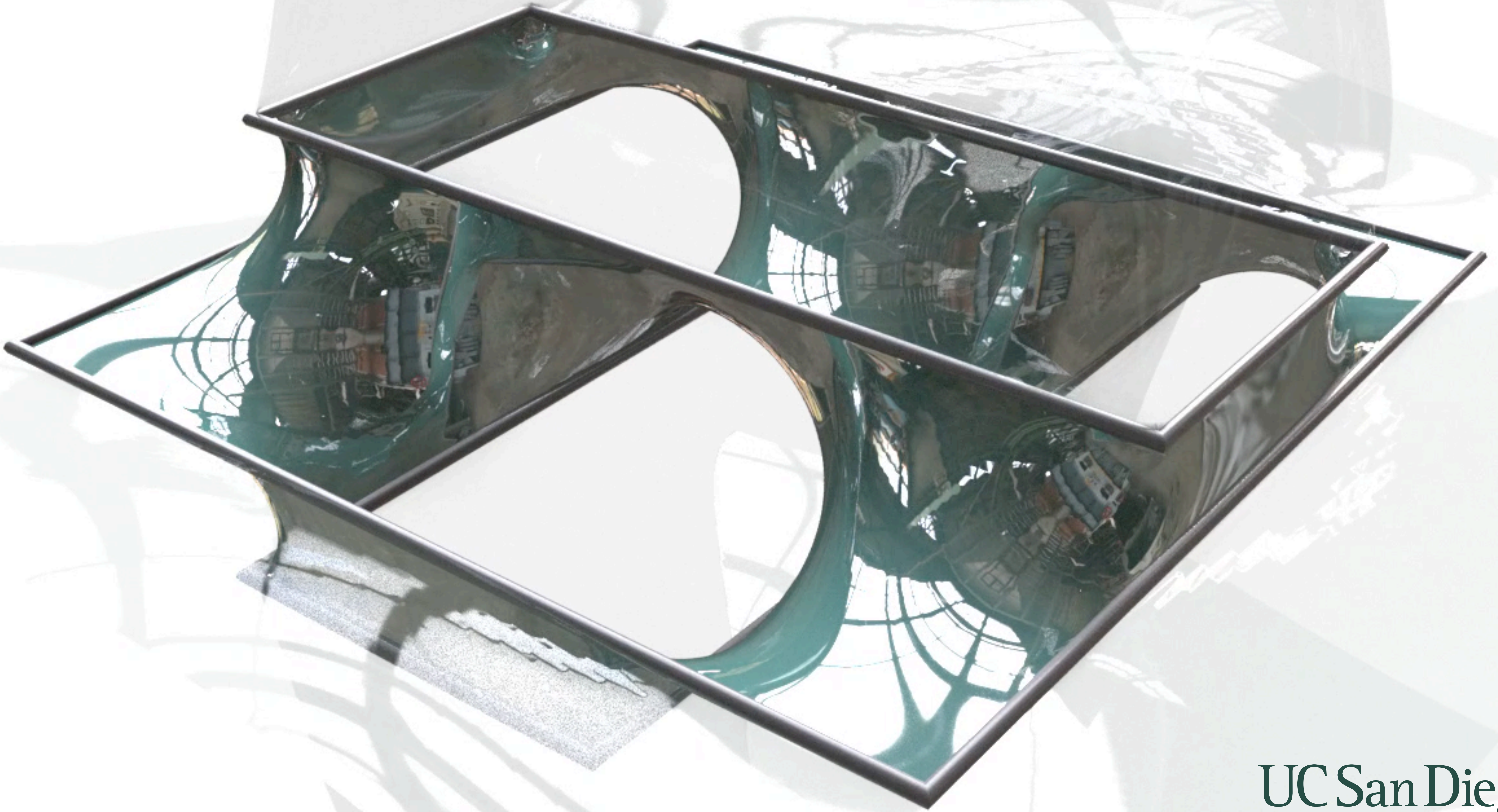

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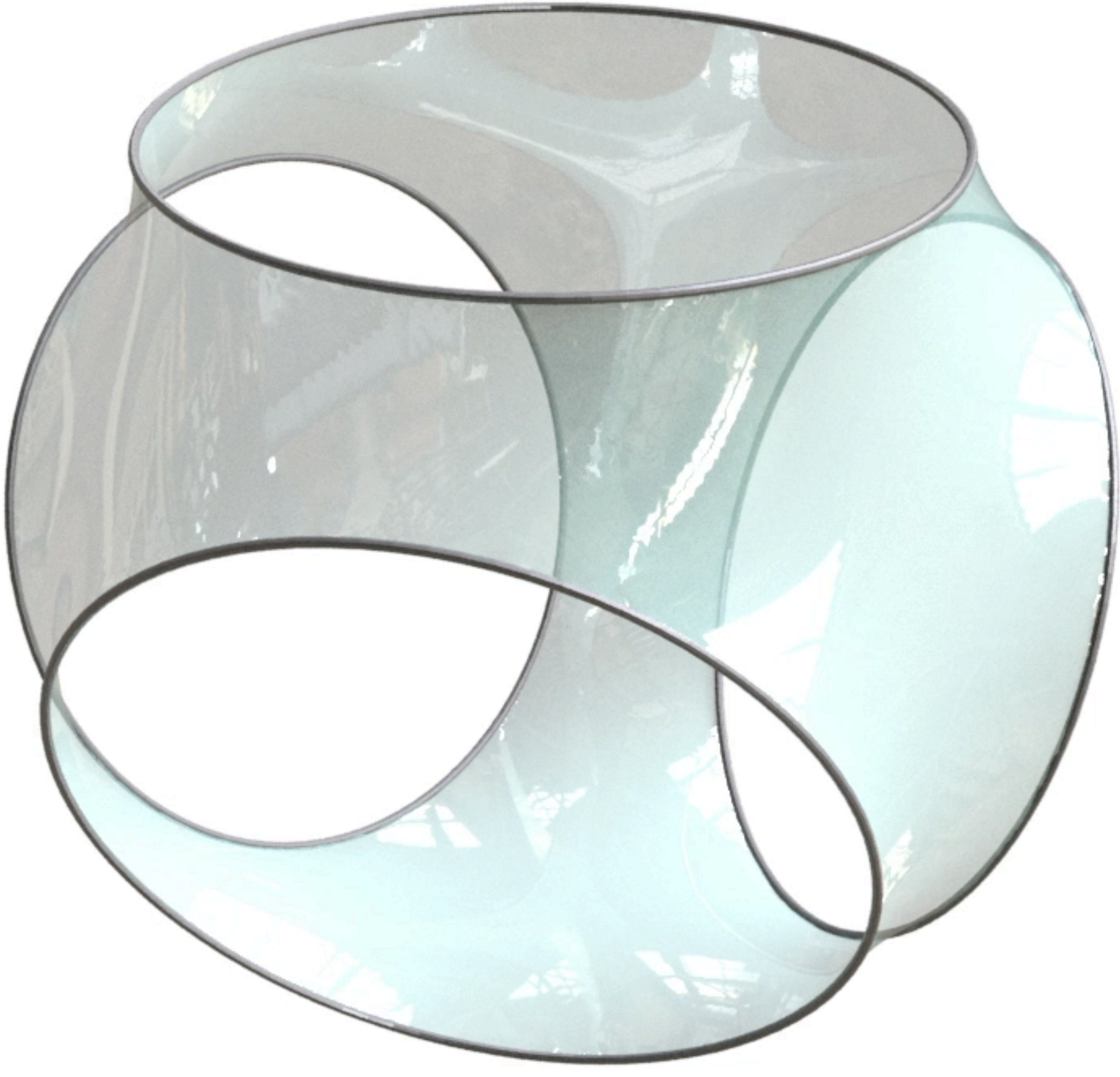

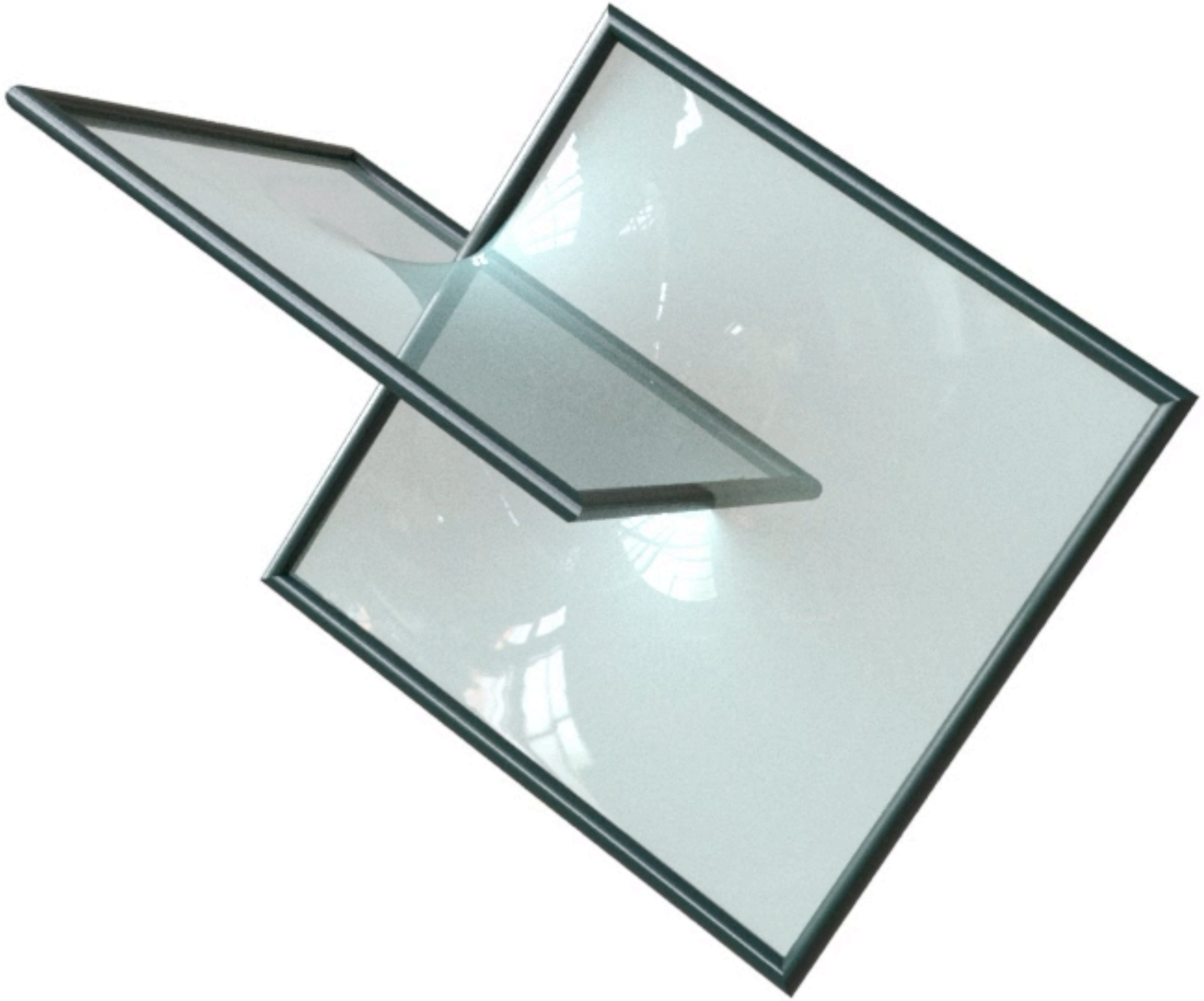

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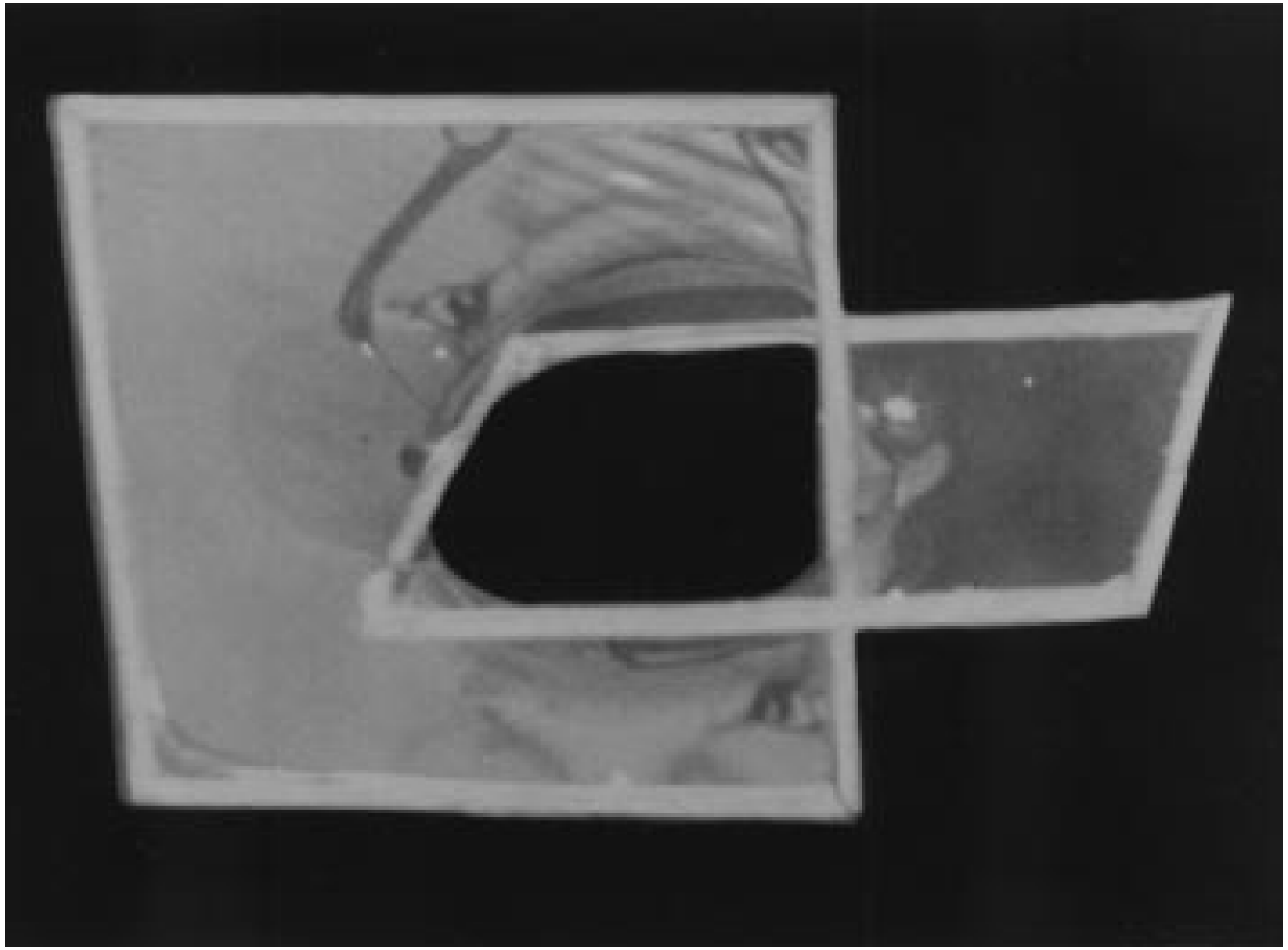
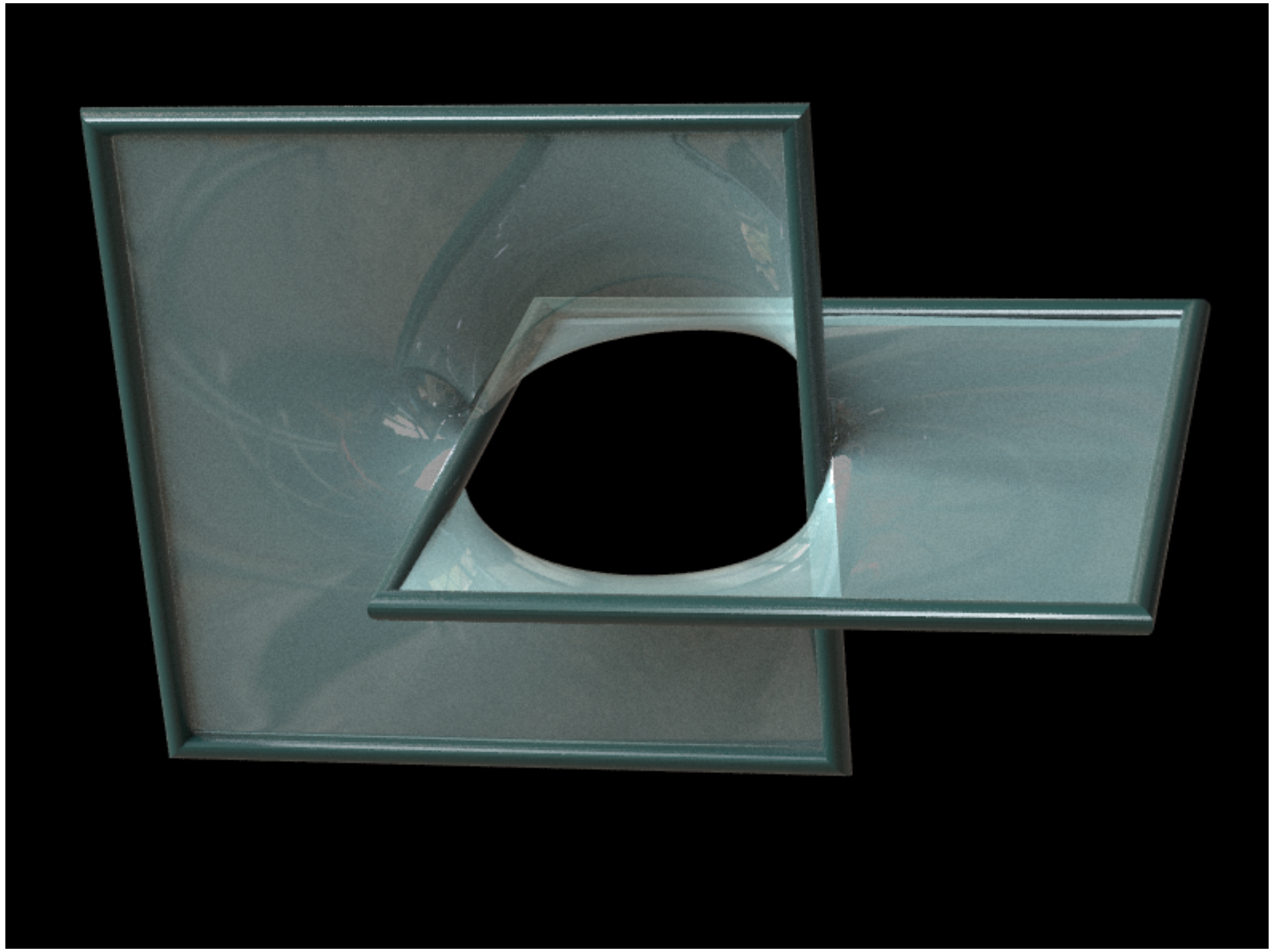
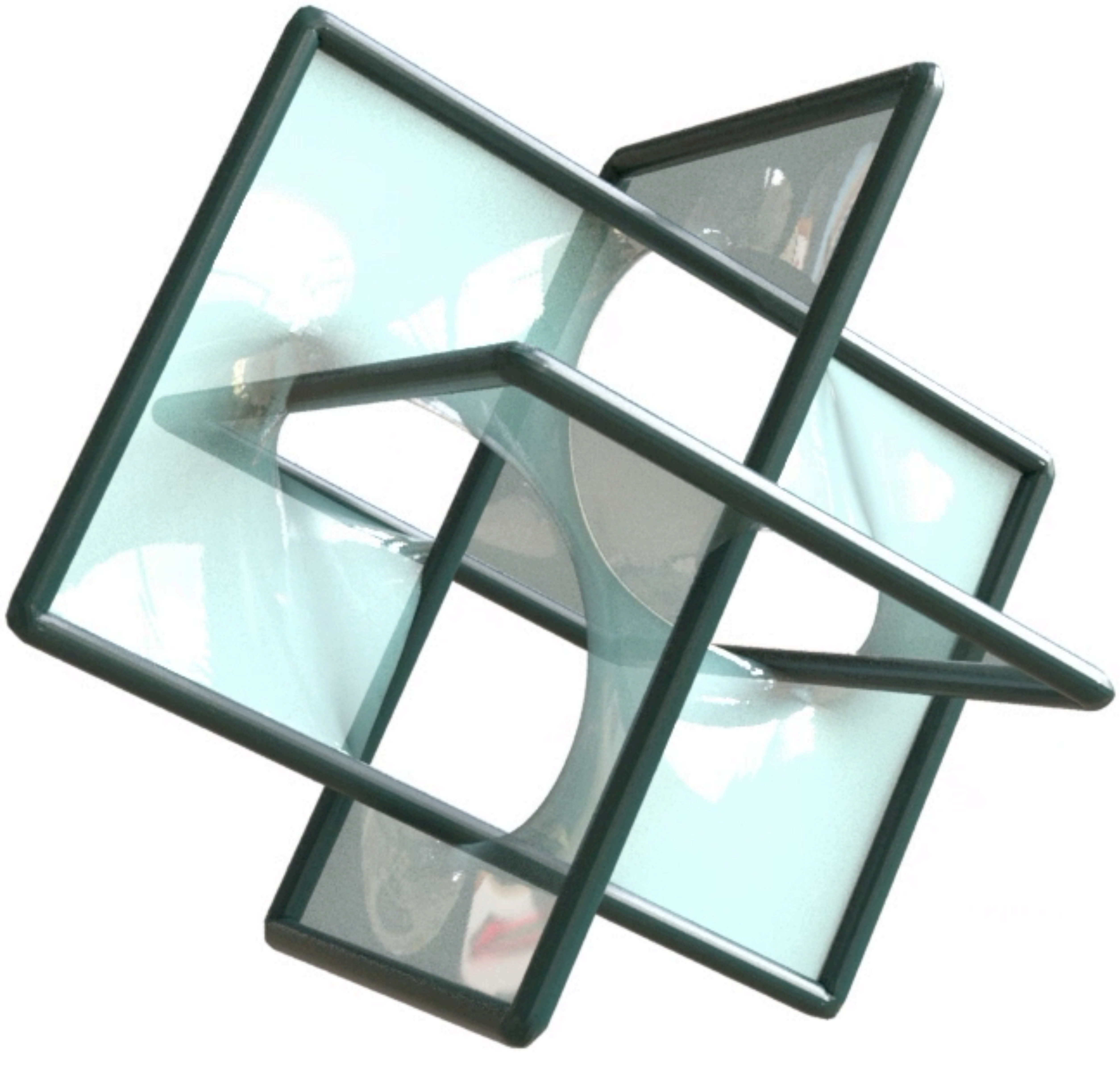
Comparison





PC: H. Parks and J. Pitts 1997



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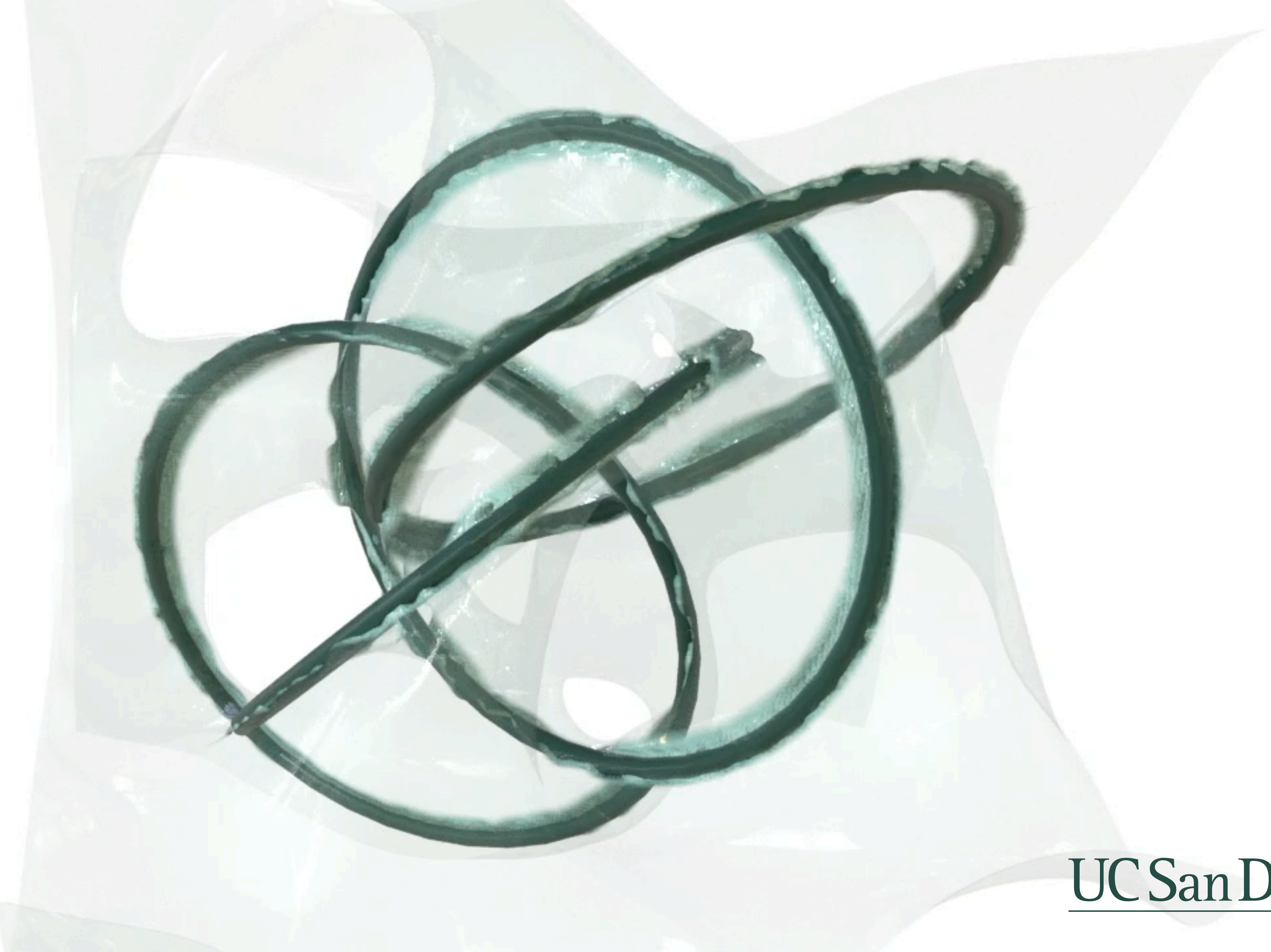

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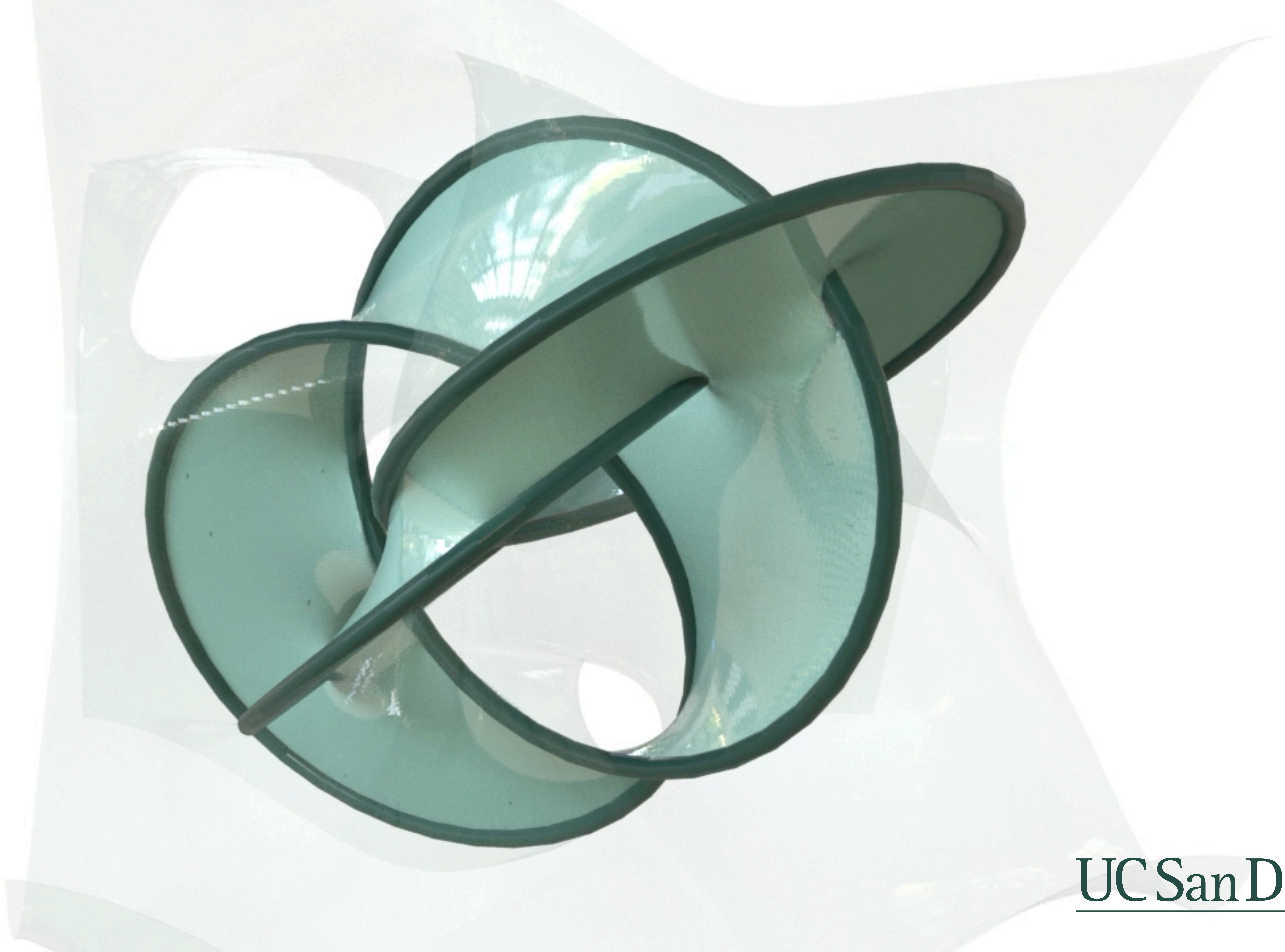

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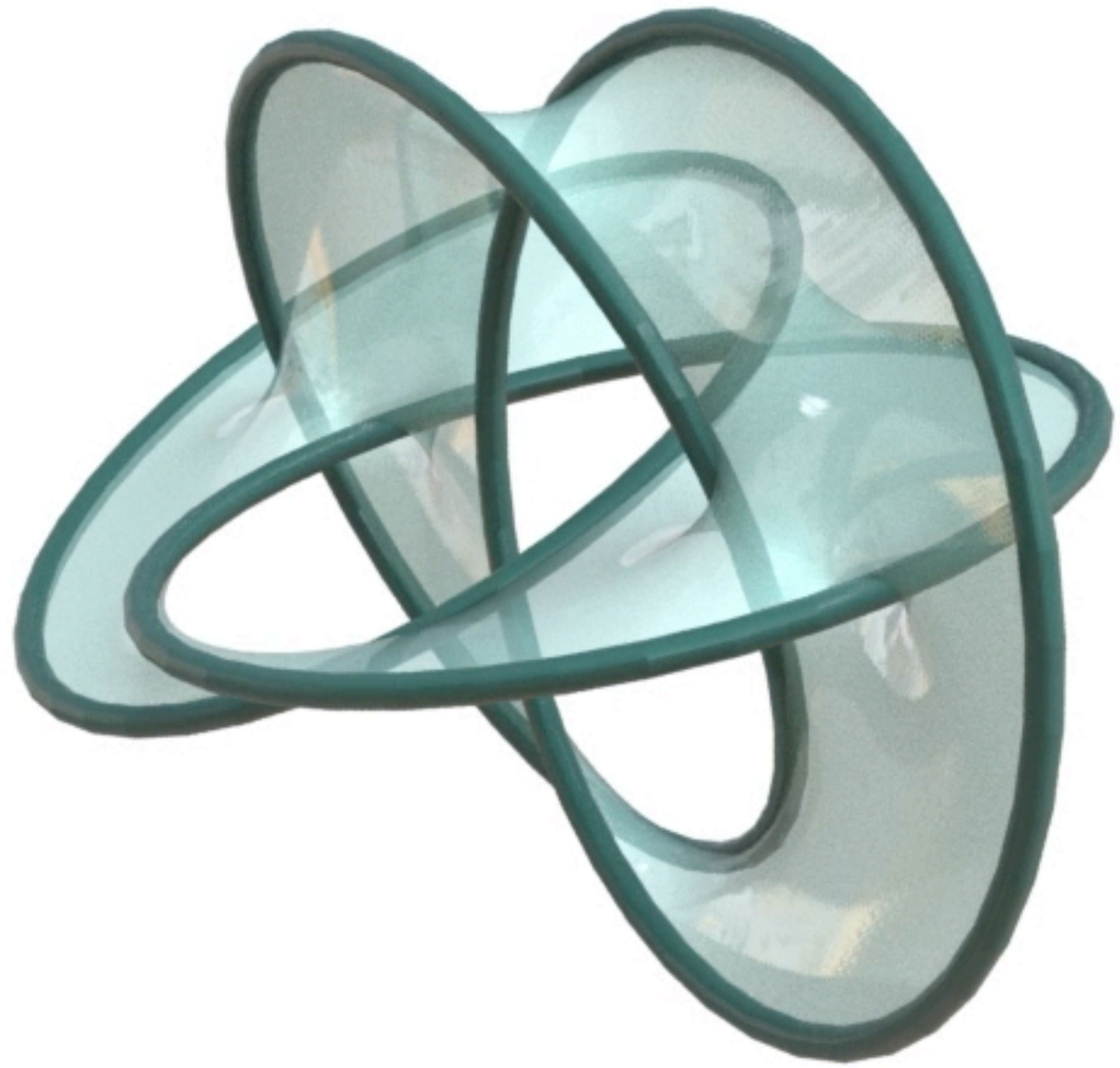

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Can’t do: non-manifold soap film

(the 120° intersection)

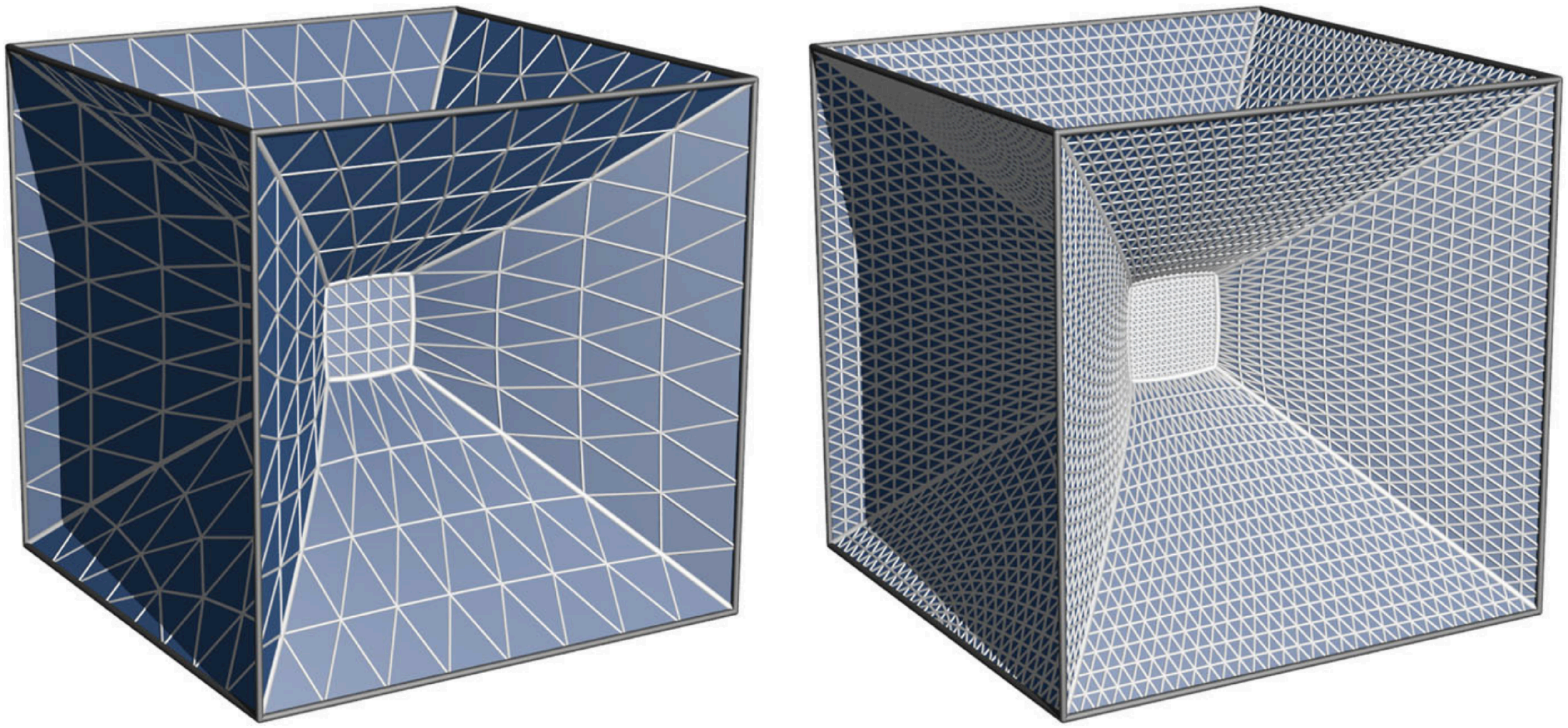

Figure courtesy: H. Schumacher and M. Wardetzky 2019

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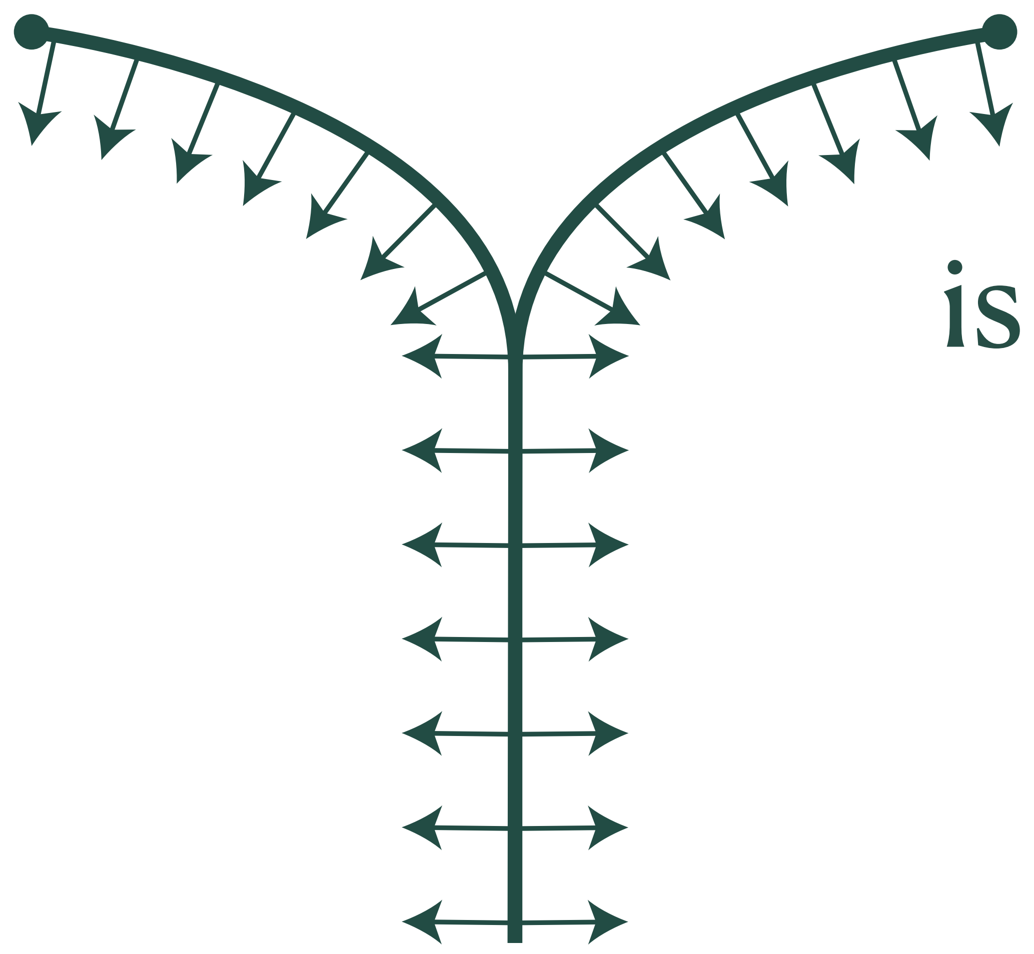
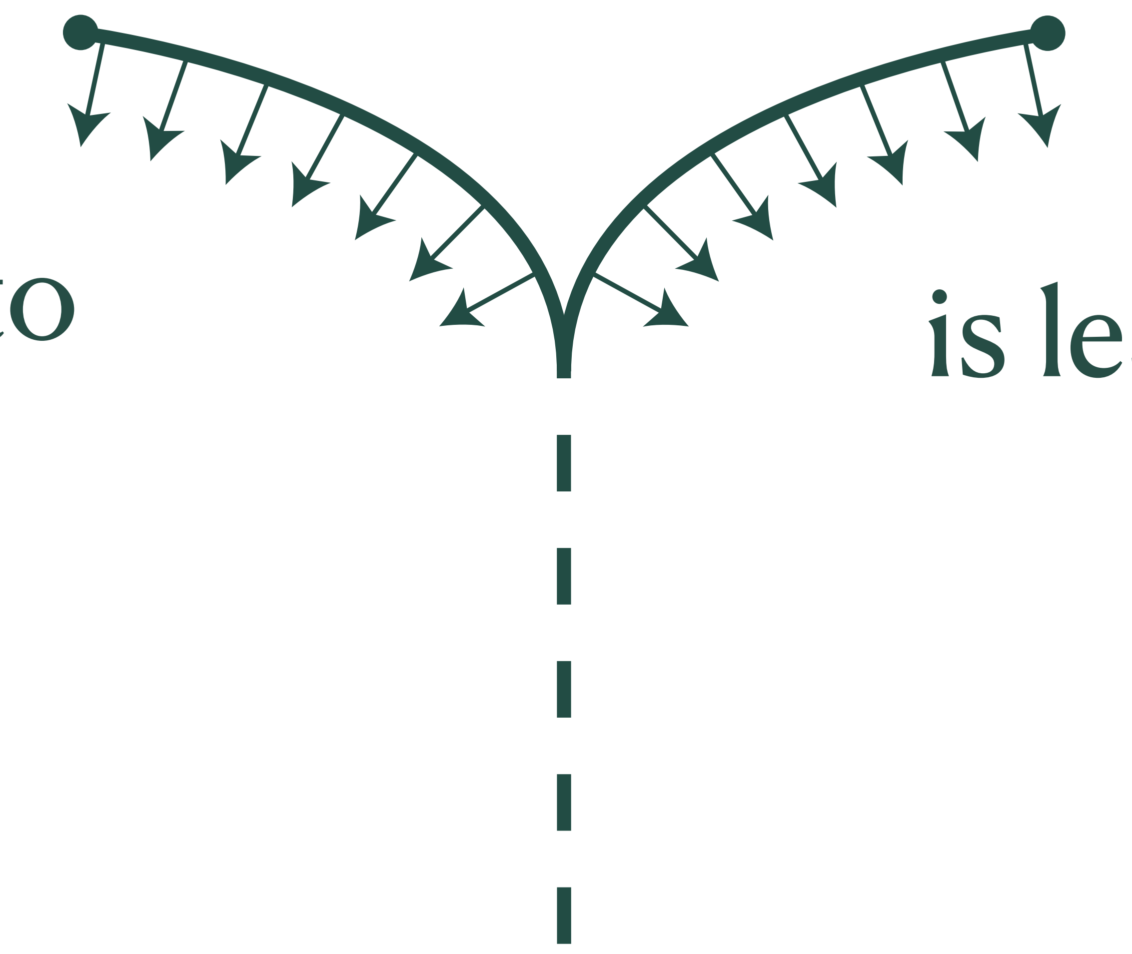
Non-manifold minimal surface

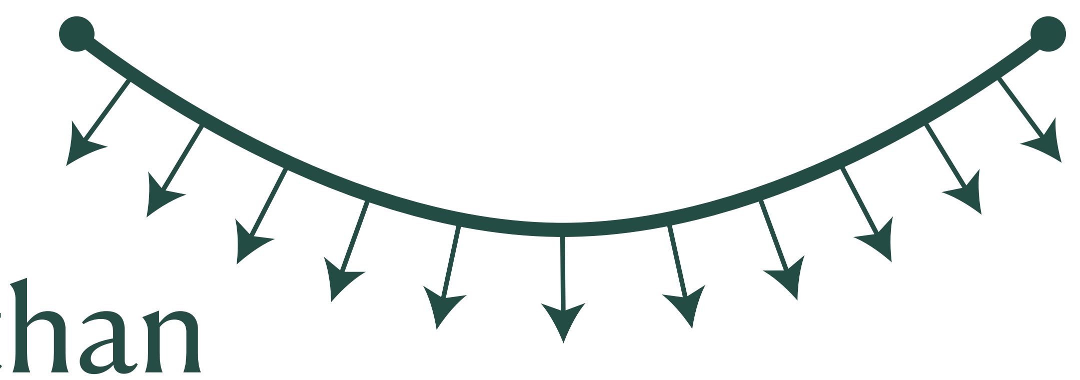
(the 120° intersection)



is equivalent to

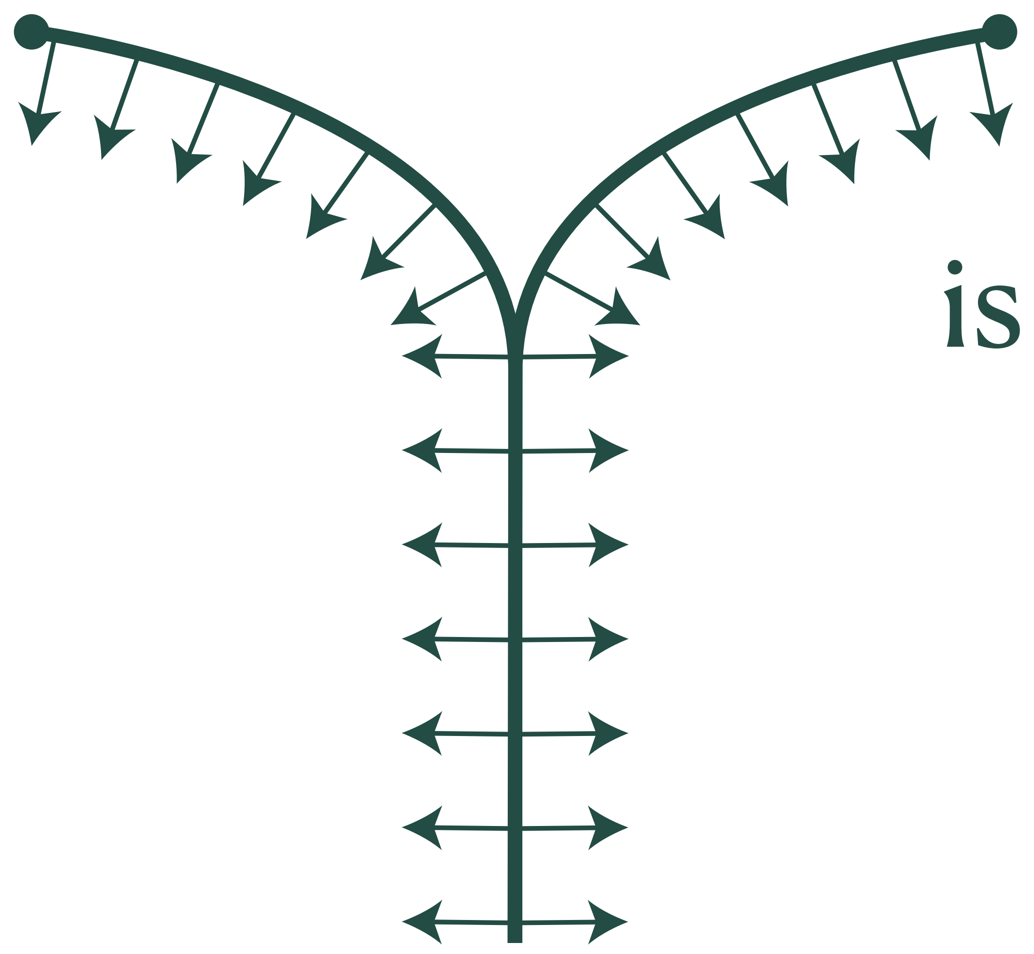


is less optimal than



Non-manifold minimal surface

(the 120° intersection)



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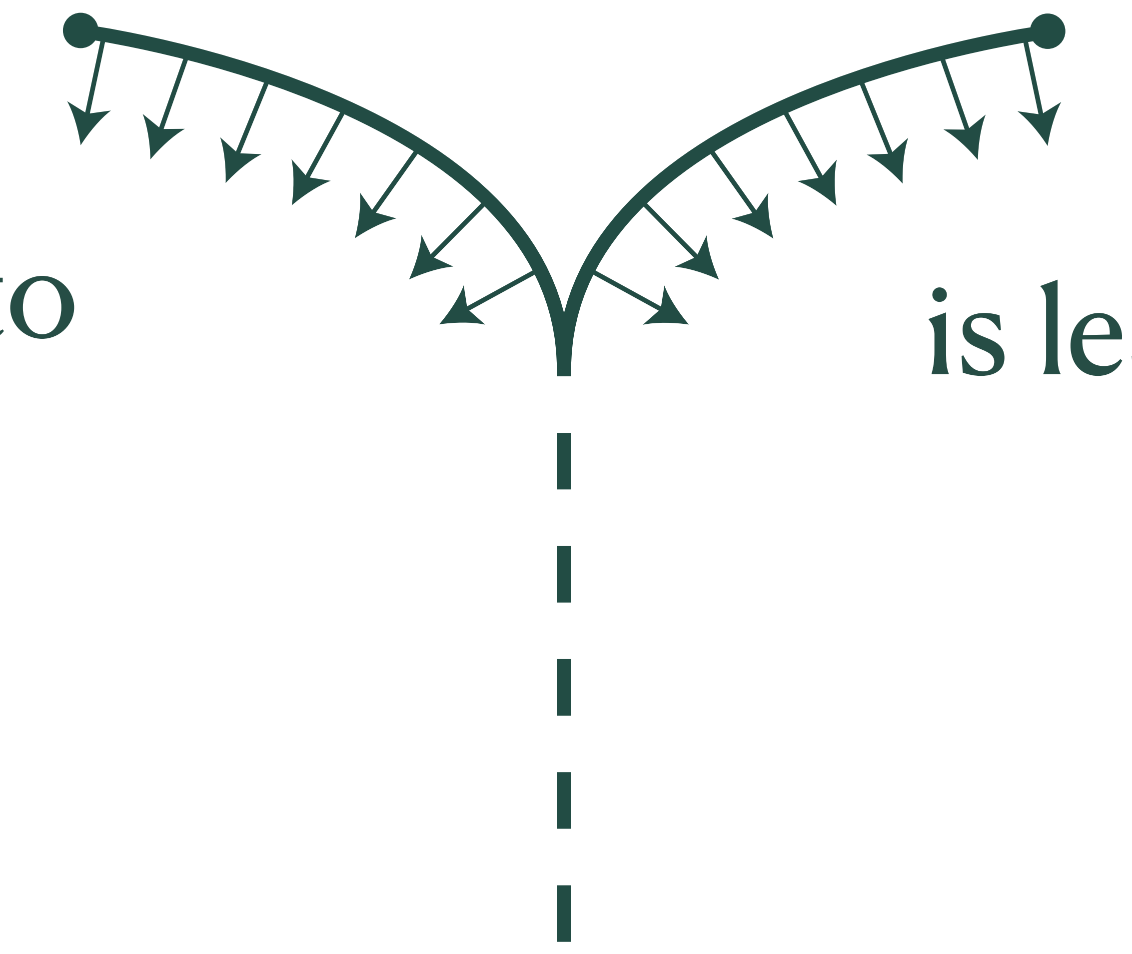

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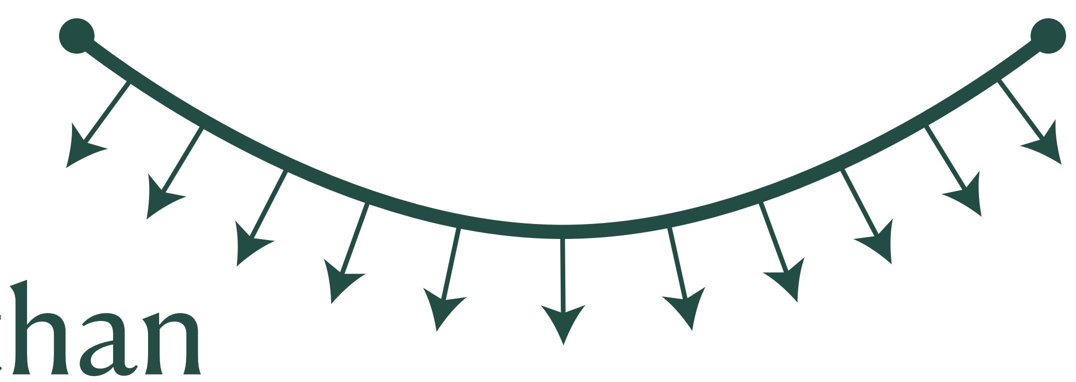
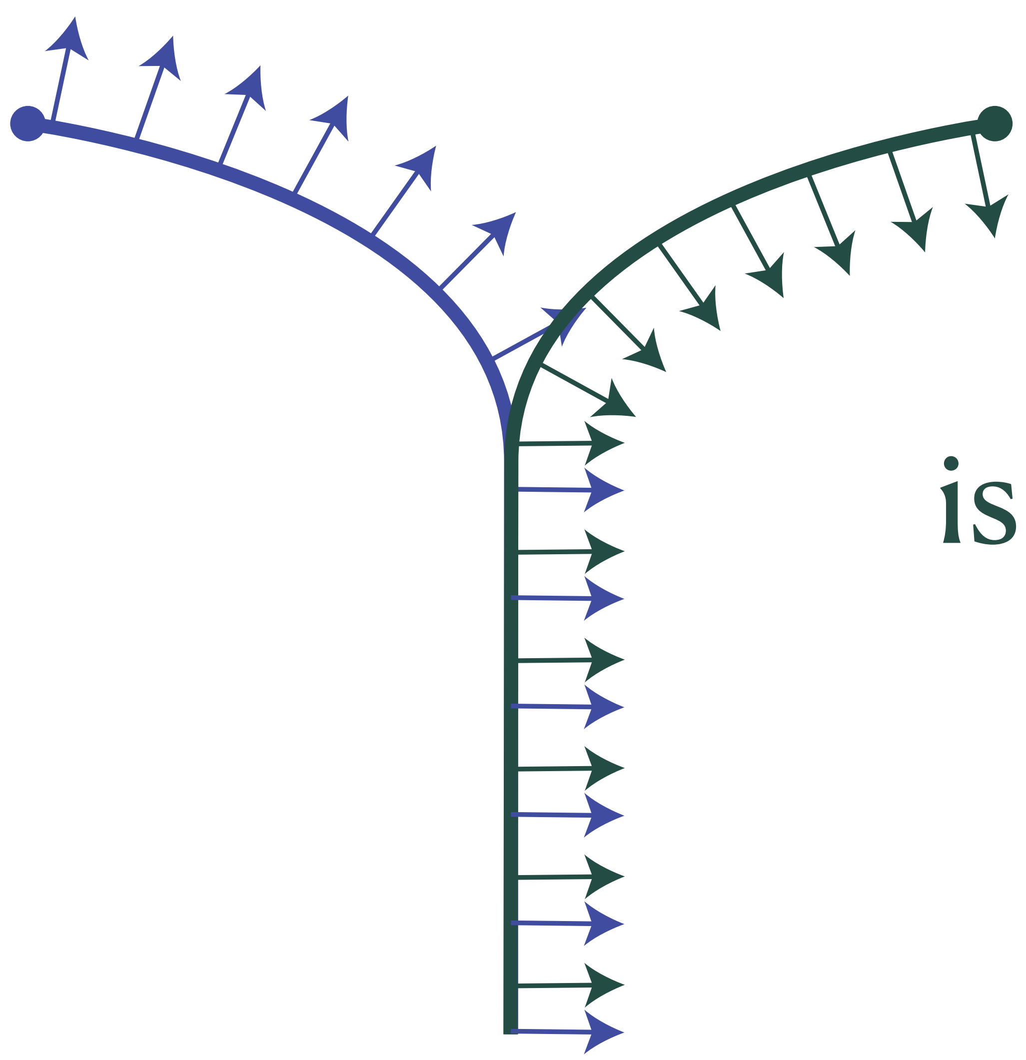
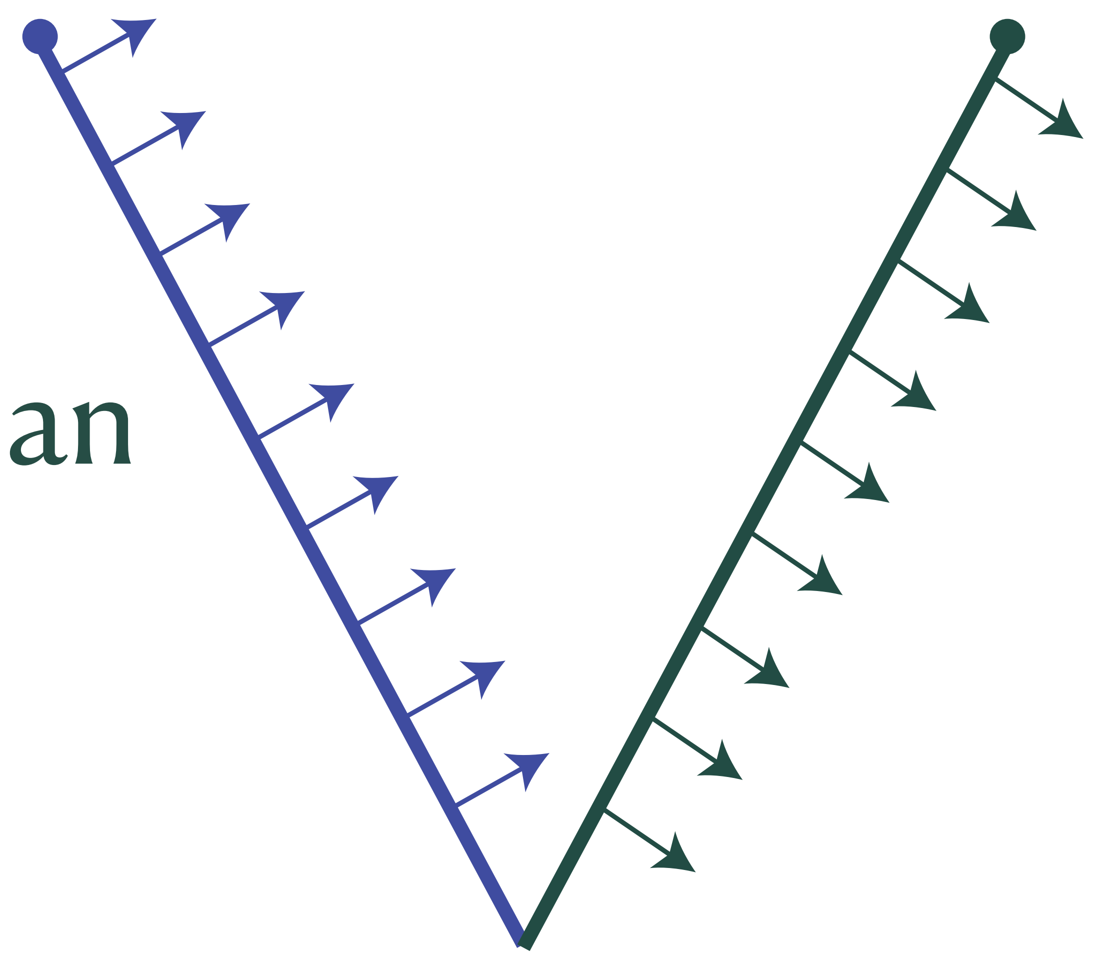




is less optimal than



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Can’t do: non-orientable soap film

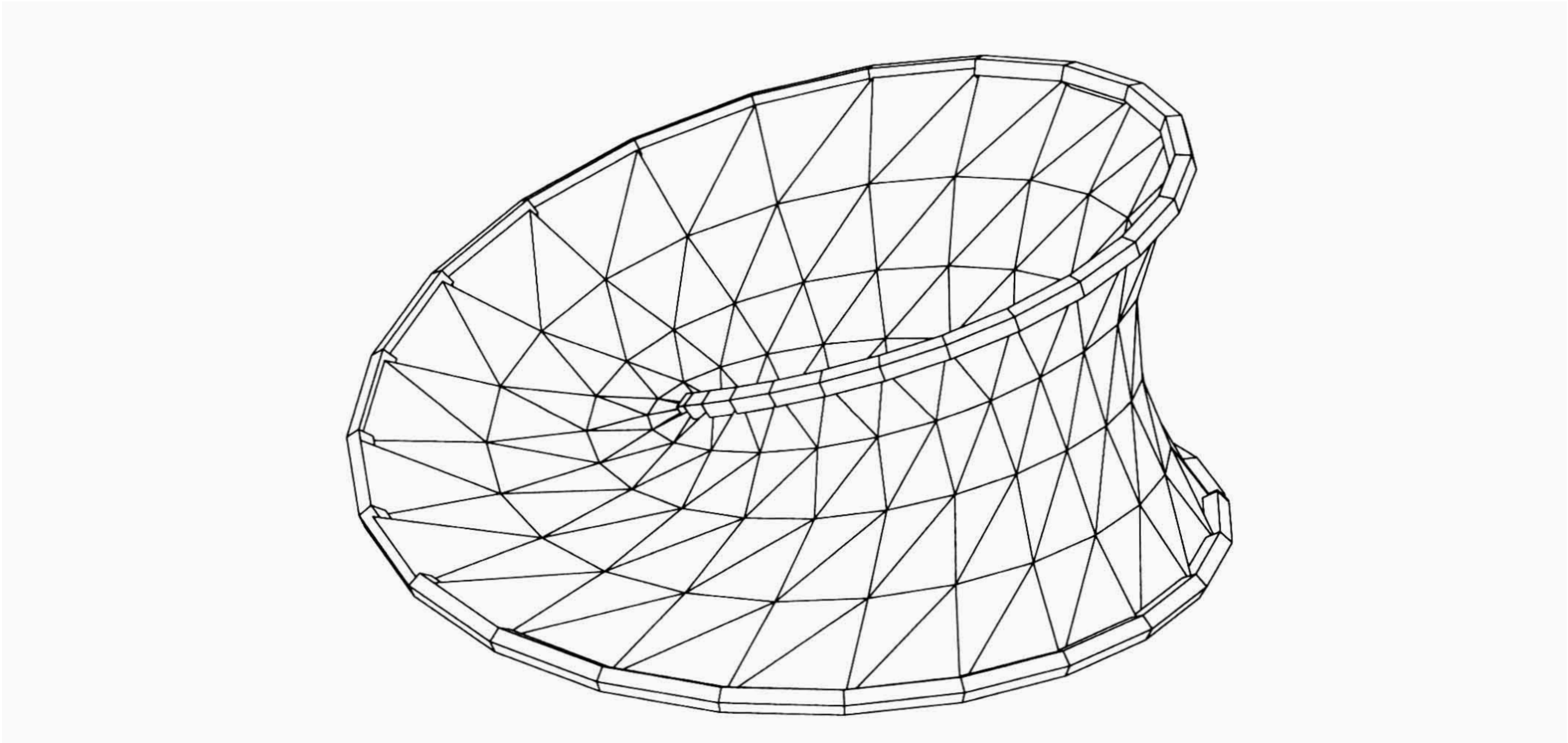

Figure courtesy: U. Pinkall and K. Polthier 1993

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Comparison to other methods

Pinkall and Polthier 1993	Dunfield and Hirani 2011	Ours
discrete curvature flow on trimesh	optimal discrete cochain on tetmesh	current norm minimization on grid
initialization/remeshing for different topology	automatic topology	automatic topology
only local minimum	global minimum	global minumum
quality surface trimesh	quality spatial tetmesh	regular grid
Possibly ill-conditioned Laplacian	Interior point method	FFT on grid

Future work

- Applications in crystallographic structures
- Current neural network
- L1 gauge decomposition of differential forms
- Gravity (soap film with mass)
- Thin shell dynamics simulation
- Non-orientable soap films using varifolds
- Volume constraint for soap bubbles
- Non-compact ambient space M
- Weierstrass-Enneper representation neural network

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Thank you for your attention!

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